PREFACE

Department of Science & Technology (DST) is promoting research and development (R&D) in frontier and emerging areas of science and technology (S&T) through its Science & Engineering Research Council (SERC). SERC is composed of eminent scientists, professionals and technologists in different disciplines of S&T, drawn from various universities, national laboratories and industries, and is assisted by a large number of discipline-specific Programme Advisory Committees (PACs). SERC has evolved, over the years, a unique and effective peer review system which has been well-recognised by the scientific community. Over the years, it has helped in promoting and strengthening several new areas of research and established a large number of national research facilities, core groups/centres for advanced research in specialised fields of Science & Technology. It has also endeavoured to promote the new concept of strengthening research capabilities in relatively small, remote and less endowed universities/departments to increase their general level of scientific standing to be at par with the national level, if not more.

The Council reviewed its activities and areas of research which had been identified earlier and decided to update them for future support. Under the overall supervision and guidance of the SERC, PACs in various disciplines had been requested to prepare a state-of-the-art document called "**Vision for R&D"** reflecting new challenges to the scientific community, national facilities required to be set up including new ways and mechanisms that may be needed for their unabated promotion.

It is with the above background that the Department of Science & Technology has decided to give wider publicity to these areas towards intensifying research and development (R&D) activities in them in future. This document **"Vision for R&D-Mathematical Sciences"** is for those are who interested in vigorously pursuing R&D in Mathematical Sciences. It is hoped that this document would be useful to the scientific community in general for planning their future research activities.

PROLOGUE

1.1 Introduction

This paper has already been published in three parts vide "Mathematics Newsletter" of the Ramanujan Mathematical Society, Vol. 6, No. 4, Vol. 7, Nos. 1&2. It presents to the readers the frontier regions of research in 10 thrust areas in Mathematics which have a wide range of applications. In this publication, we append this prospective note by DST's first ever Programme Advisory Committee on Mathematical Sciences for provoking a thought process, by the community of mathematicians at large, far and wide in the country, by which DST wishes to benefit in its future promotional efforts.

The Programme Advisory Committee on Mathematical Sciences (henceforth referred to as PAC- MS) of DST went through the following steps in the preparation of the vision paper.

1. Identification of ten thrust areas in Mathematical Sciences.
2. Members of the Committee were assigned the task of preparing a brief write-up on each of the topics.
3. These papers were discussed at several subsequent PAC-MS meetings, modifications effected and then rewritten.
4. Subsequently a special meeting was held in Delhi, for discussions on these write-ups, between the PAC-MS, a selection of invited mathematicians, theoretical computer scientists, physicists and other experts as well as a few representatives of the corporate sector/industry. What follows is the final version of the document integrating the views of all these sections gathered through all the above processes.

During this process, the PAC had to bear in mind its terms of reference at the time of its creation, viz, development and support of a multidisciplinary approach in mathematics. This, of course, gets reflected to some extent in the choice of the ten thrust areas. It is also understandable that the personal expertise and preferences of the PAC members will be reflected in these choices. As the PAC's are reconstituted once in about 3 years, this approach would eventually represent interests of the Mathematical Sciences community at large. On the other hand, this PAC time and again reiterated the basic principle that there should be some **non-trivial mathematical content** in the projects that it would like to support, or in other words routine applications of well-rehearsed mathematical methods are not to be encouraged in most cases. However, even in these cases where a routine methodology may lead to something significant in application, mere potentiality or possibility is not considered enough and the PAC looks for some concrete evidence of interest on the part of the deemed end-user(s).

1.2 Thrust Areas

For the purpose of intensive support, possibly even proactively, the following ten areas have been identified by the PAC :

i. Computational Aspets of Geometry and Algebra
ii. Numerical Schemes and Qualitative Properties of Solutions of Differential Equations.
iii. Stochastic Process Modelling
iv. Deterministic Control Theory
v. Exploratory Data Analysis
vi. Dynamical Systems
vii. Game Theory
viii. Combinatorial Optimization
ix. Spectral and Inverse Spectral Theory
x. Wavelet Analysis.

At the very outset one must emphasize that this does not mean that the PAC has not supported other areas of Mathematical Sciences. These are the areas where some special efforts are needed for their development in the country. In other words, the list is only indicative and not exhaustive. It is expected that there will be considerable overlap between some of the thrust areas and other areas of Mathematical

Sciences. In any case, the PAC looks at all project proposals independent of these thrust areas, but, of course, each is evaluated on its individual merit.

It is the view of the PAC that in some of the areas listed above, e.g. (vi) and (vii), there is very limited front-level research expertise in the country, particularly if one emphasizes the mathematical aspects. However, there are some resources to start basic and middle-level training programmes. It should be added here that the PAC-MS has an additional mandate to recommend funding of awareness programmes, training schools and workshops on certain selected topics in Mathematics. It is felt that this should be one of the priorities to identify resource persons in an area, organise training schools or workshops at various levels, or one at a multi-level, have the lecturers prepare lecture notes and disseminate these widely, especially in thrust areas. This approach, it is hoped, will eventually generate necessary expertise in the country on the trust areas.

Broadly speaking, Mathematics enjoys and has a dual role to play in various areas of human endeavour. Firstly, Mathematics is one of the finest products of human intellect. Secondly, many areas of Mathematics find very striking applications in diverse fields ranging from the Natural Sciences to Economics. In fact, the remarkable success of Mathematics in so many fields appears to be more than what one has any reasonable right to expect. From antiquity till now there has also been a kind of feedback between these two roles of Mathematics, i.e., some of the ideas from the other allied fields have given rise to beautiful new structures in the purer form of Mathematics. This interplay of the two roles speaks of the universality of Mathematics as a language of Science and has to be actively encouraged.

In the Indian context at present, there are two main agencies supporting research and training in higher Mathematics :- The National Board for Higher Mathematics, (NBHM) under the Department of Atomic Energy (DAE) and the DST, though of course there are others like Council of Scientific & Industrial Research (CSIR) and University Grants Commission (UGC) which also support the cause of Mathematical Sciences. It should be mentioned at the outset that for quite some years the NBHM has been actively and successfully running the Indian Mathematical Olympiad (IMO) and Mathematics Teaching and Training School (MTTS) programmes for the school-leaving and undergraduate students. It is envisaged that these two agencies would work in close co-operation and would amply try to complement each other's support activity.

The development of Mathematics in India has been starkly uneven. A few Indian mathematicians have, over the last few years, made research contributions at very high levels and there exist a small number of research centres/universities that have a reasonable level of international repute. However, the general level of teaching and research in Mathematics remains very poor and there are large number of areas of modern Mathematics with hardly any expertise in the country. Though the syllabi of Mathematics in the universities is not the direct concern of this PAC, it must be said that they often include out-dated topics as well as too specialised ones at the expense of some of the more essential ones. It has been felt that this has led to some amount of lop-sided training of many of our students as they themselves discover when they enroll for research. Though this PAC cannot make a direct impact in this matter, it can certainly help by supporting good quality short-term training programmes where a seleted number of teachers from colleges and universities may be invited to participate.

The experience of this PAC has been that the number of good quality research proposals in Mathematics is small, as compared for example to those in Theoretical Physics or Chemistry. It appears that at most institutions of some quality, the research needs of mathematicians are covered to a fair extent. One feels that there has to be more of active good mathematicians, in small groups, in many more places before a steady stream of high quality research proposals start coming in.

No discussion on the state of any programme/initiative in the country is complete without a critical look at the job market for the products of the initiative. But nevertheless, everyone surely recognizes the fact that if there are not enough jobs created with reasonably good facilities, the brighter- students are just not going to be attracted to a career in Mathematics. There can be a three-pronged attack on this problem.

(a) Create a few, say 20 to start with, Research Associateships for 5 years (with financial compensation comparable to that of a Reader in a university) to provide young post-doctoral researchers a few years of stable research enviornment as well as to give them a bit of 'breathing space' while they look for a more permanent job.

(b) Consider creating one or two small centres for high quality research in Mathematics, closely allied with a university. In few of them often experienced difficulty in maintaining academic standards and efficient running of many centres/institutes. This PAC is a bit wary of this aspect. Nevertheless this is a possible approach which should be explored by the PAC keeping in mind that there should be no compromise with the quality of the personnel involved.

(c) Initiate a dialogue between a selected group of mathematicians and some representatives of the Indian industry to impress on the one hand upon the Industry that with the integration of modern computers with modern industry in all sectors, the role of Mathematics in industry is ever increasing worldwide and on the other hand to inculcate a "hands on problem-solving" attitude amongst the mathematicians. For this, the PAC may support a few meetings of the above type, provided it is convinced that there is enough enthusiasm from both sides and there is evidence of some pre-meeting preparations. It should be emphasized that the decision-makers in the industry, not being with an academic background, would need convincing reasons to advance the cause of Mathematics. With the changes in the economic scenario in the country, mere problem-solving may not anymore be the focussed aim of the industry, instead it may be looking for original/innovative product development. We may be far from having a research laboratory (employing mathematicians) of the level of IBM or Rand Corporation etc., but a small beginning can be made by starting these dialogues. Only after such initiatives have yielded some tangible results for the industry, one can approach the industry to sponsor some amount of 'blue-sky' research. As the Indian industry matures in its outlook it will probably start looking at some of these aspects with greater depth and emphasis.

To summarize, aside from the evaluations of all project proposals in Mathematics, the PAC-MS would like to encourage the following:

- Holding of SERC schools in the thrust areas at various levels with usually one or two lecturers (to be identified) lecturing over a week or more on clearly defined topics and providing lecture notes.

- Organizing workshops/training schools in various areas of Mathematics and allied fields (over and above those referred to above) with special emphasis on the participation by college/ university teachers/students.

- To choose a maximum of five promising participants, identified by the director and the faculty of the above mentioned schools/ workshops for support for a long term (a few months) visit to the relevant experts, institutions to encourage continued interaction subsequently.

- Inviting proposals from colleges/universities/institutes for funding short visits to the respective institutions by experts in the thrust areas for interaction and collaboration.

- Holding inter-disciplinary meetings between mathematicians, practitioners of the other sciences and technology fields where mathematics is used, and the industry.

- Creation of one or two small centres of excellence in mathematical training and research, preferably in close association with a university.

- Supporting a limited number of scientists with a strong track record of high quality research in Mathematics with flexible research grants which can be used for their travel to conferences and other centres in India, for inviting other scienists for collaborative purposes and for contingency expenditure.

- Organising exhibitions and competitions in Mathematical Sciences to spread Mathematical culture and awareness in the country, and to excavate talent.

- Creation of 20 positions of Research Associates in Mathematical Sciences (with particular emphasis on the thrust areas) for a period of five years with financial compensation comparable to that of the Readers in a University to be associated with selected universities/institutes.

- Creation of Advanced Lecture Circuits in Mathematics, perhaps modelled after a similar one in Physics run by DST.

- Support for the libraries in selected universities/institutes (in particular for promotion of research in less endowed universities), to enable them to procure books and journals in Mathematics.

COMPUTATIONAL ASPECTS OF GEOMETRY AND ALGEBRA

2.1 Introduction

Computational problems in Geometry and Algebra have two distinctive flavours: discrete problems that arise typically in Computer

Science research and problems in a continuous setting that arise in Mathematics and areas of its applications.

2.2 Computational Problems in a Discrete Setting

Discrete Computational Geometry is a very active area of research, the main focus being the design of algorithms for geometric problems. Typical problems are constructing the convex hull of a given set of points, and finding the Vornoi regions for a given set of points. Although very well studied in two dimensions, the design of efficient algorithms for higher dimensions is open. Another area of interest with many practical applications is computing regions of visibility in a domain with or without obstacles. Questions such as the minimum number of guards to be placed in a region, so that they cover the entire domain, have an important bearing on certain optimization problems especially using the notion of domination in graphs.

Combinatorial Geometry is concerned with understanding the properties of points of configurations consisting of points, lines, planes, hyperspaces, etc. Typical questions are : whether a certain configuration of points is realizable in space; if so, what is the least dimension of the space in which it can be realized; if not, a proof of the non-realizability. Such questions arise in trying to show lower bounds on the complexity of computing functions.

Geometric techniques with a strong computational flavour also play an important role in solving optimization problems. Perhaps the best evidence of this are the polynomial time algorithms for solving Linear Programming problems discovered in the past 15 years. Tools like projective transformation of linear equations and basis reduction have lead to efficient solutions to linear programs and better approximations to classical diophantine equation problems. The study of polyhedra, that arise in optimization problems, Polyhedral combinatorics, the questions that characterize them, equations as to whether they have integral points etc., have contributed in a large way to our understanding of optimization problems.

2.3 Computational Problems in a Continuous Setting

Objects of interest here are usually described by a set of polynomial equations and polynomial inequalities.

Computer Graphics is primarily concerned with displaying objects on a computer screen. The issues involved are many : modelling the object so that the model is simple but still retains information of its complexities-for example, its singularities. Data represention and storage, algorithms for generating perspectives and for displaying hidden surfaces, are some of the important considerations. Solutions to similar problems in a discrete setting find their use to solve such problems. In the recent past non-trivial algebraic geometric concepts like approximations using semi-algebraic sets have been used in real geometric modelling.

Symbolic Computation as a means of solving complicated geometric and algebraic problems is gaining importance. Computer-aided tools such as Macaulay and Maple are classic examples of tools developed primarily for the purpose of being able to compute invariants of varieties defined by polynomial equations. This area has far reaching applications in Physics, Mathematics and Engineering. The theory of Gröbner basis forms the cornerstone of this area.

Computational Topology is a fast emerging area. The main thrust here is to be able to efficiently compute topological invariants of objects. The emphasis is to develop algorithms that are more geometric in nature and avoid using symbolic computations which are often inefficient.

The study of invariants of algebraic varieties is gaining importance in Computational Complexity theory. Recent results on lower bounds for computing certain Boolean functions in weak models, have been based on the inability to approximate these functions by low degree polynomials. It turns out that high values of Hibert function (an important invariant of varieties) imply such non-approximability results. Indeed it is believed that very difficult lower bound results in Computational Complexity theory which seem to be inaccessible via techniques that are currently available might be obtainable by reducing the concerned questions to an appropriate algebraic geometric setting.

At present the following institutions are amongst those that are active in some of the areas mentioned above : the Departments of Computer Science in the Indian Institutes of Technology (IITs) at Mumbai, Kanpur, Delhi and Chennai, and in the Indian Institute of Science (IISc), Bangalore, the Computer Science groups of T.I.F.R., Mumbai, The Institute of Mathematical Sciences (IMSc), Chennai and the Chennai Mathematical Institute.

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NUMERIAL SCHEMES AND QUALITATIVE PROPERTIES OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

3.1 Introduction

Ordinary Differential Equations (ODE) and Partial Differential Equations (PDE) arise in models of various situations (physical, chemical, engineering , etc.). Solving them is imperative to understand various phenomena which they describe. Looking into the history, we notice that four broad approaches have been followed to achieve this goal, viz. (i) obtaining explicit solutions by some ingenious ways, (ii) obtaining

asymptotic solutions by the application of a wide spectrum of techniques from applied mathematics, (iii) physical experiments, and (iv) numerical experiments.

Electronic machines offer an alternative to physical experiments which have become either expensive or impossible at times. As prophesied by words of John von Neumann "really efficient high speed computing devices may, in the field of nonlinear PDE's as well as in many other fields which are now difficult or entirely denied of access, provide us with those heuristic hints which are needed in all parts of mathematics for genuine progress.... many branches of both pure and applied mathematics are in great need of computing instruments to break the present stalemate created by the failure of the purely analytical approach to nonlinear problems".

Having mentioned the importance of computation, let us cite a few remarkable achievements.

Fermi, Pasta and Ulam discovered the remarkable, almost periodic, behaviour of the vibrations of nonlinear chains and Kruskal and Zabusky, the generation and interaction of solutions. The complete integrability of the Tada Lattice became plausible through very careful numerical calculations by Joe Ford. Mitchell Feigenbaum discovered the remarkable universal laws on iterations by analyzing numerical experiments. Numerical studies led Lorenz to discover the concept of a strange attractor. The understanding of chaotic behaviour of simple dynamic system coexisting with islands of stability has been enhanced by numerical studies.

Our objective here is to examine the various issues involved in this kind of calculations, highlight the progress made and say a few words about the future course of action.

The following are the steps which are usually involved in the computation of solutions of ODE/PDE : discretization, solving discrete equations, convergence analysis, acceleration procedures, etc. In the sequel, we will discuss these steps one by one.

3.2 Discretization

The rapid progress in computing techniques was made possible by striking improvements and novel ideas introduced in the discretization of the equations. Let us name a few commonly used techniques.

(i) Finite Difference Method (FDM)
(ii) Finite Element Method (FEM)
(iii) Spectral Method (SM)
(iv) Wavelet Method (WM)
(v) Particle Method, Vortex Method, etc.
(vi) Finite Volume Method (FVM)
(vii) Variational Methods, Optimization and Mini-max Algorithms
(viii) Nonlinear Galerkin Methods (NGM)
(ix) Other Methods : There are other techniques which combine the ones listed earlier and which are based on different formulations of the problem. Let us cite a few : domain decompositions method, parallel computing, nonlinear least squares, operator decomposition, multigrid method, adaptive methods etc. While modelling physical processes some parameters may be neglected. One way of suggesting an approximation involves the reintroduction of the parameters and this process is called unfolding. Novel schemes are born this way. An example is the mean fee path which gives rise to kinetic schemes.

3.3 Solving Discrete Equations

We are not going to dwell on this point in this brief write-up. We merely point out that there are plenty of clever algorithms, both direct and iterative, to solve the resulting (nonlinear) algebraic equations. One of the main concerns is to minimize the number of operations, computer time, etc. One has to devise ways of exploiting the sparseness of the system which is usually large. We should also worry about the number of conditions of the system. This is essential if we want to check the growth of round-off errors.

3.4 Convergence

A straightforward discretization of given equations may not converge at all. Even if it does, it may converge to a wrong limit. This is especially true in the case of nonlinear equations which possess multiple solutions and other instabilities. Therefore the question arises; how to believe the numbers churned out by the machine? In other words, is there convergence? if so, how to accelerate it at minimal increase in the cost? Can one estimate the error? These are some of the issues which we take up now.

The answer to the question of convergence is provided by the Central Theorem of Numerical Analysis which states that a consistent and stable scheme is convergent. In general, stability will depend on the a priori estimates of the exact solution. These estimates usually imply some weak convergence for a subsequence which is enough to pass to the limit in linear problems. In the case of nonlinear problems, the above weak convergence for a subsequence which is enough to pass to the limit in linear problems may not suffice. This is simply a manifestation of instabilities created by nonlinearities. To overcome this, we require some compactness criteria of Rellich type. The 'Compensated Compactness' result of Murat-Tartar is a powerful generalization of Rellich's Theorem.

Admitting convergence, the next step is to obtain error estimates. It is well-known that the error depends not only on the order of the consistency but also regularity of the exact solution. This explains the success of FEM in the case of linear elliptic problems and the difficulties in the case of turbulent fluid flows where the velocity field is irregular.

So far we have been discussing approximation of well-posed problems. Let us now focus our attention to some singular problems which arise in practice. One of the open problems is to suggest schemes which can efficiently compute irregular solutions. One idea to overcome this difficulty is to know the location and the nature of the singularities of the solution. (Mandelbrot's seminal observation about the fractal character of the singularities of the velocity field in fluid flows requires an explanation). Incorporating these singularities into our approximation scheme, we will be able to compute 'rough' solutions. In order to localize the singularities, pointwise estimates will be of immense help. Alternatively one has to obtain the decay rate of wavelet coefficients. These are programmes for the future. Thus we see how qualitative properties of the solution are intimately connected with its approximation.

3.5 Other Issues

So far we have been considering direct approximation of a given model. In some cases, it is possible to 'simplify' it before proceeding to make computations. Now we will look at some examples in which once again qualitative properties of the solution play crucial role. If we have oscillating coefficients, domains or boundaries, we can use Homogenization to produce simpler models. The same idea can be used to treat higher Fourier modes and to set-up a turbulence model. In spite of having several tools like H-convergence, R-convergence and several measures to quantify oscillations and concentrations present in the solution, justification of a turbulence model has not yet been achieved.

Similarly, if one has three dimensional bodies which are thin in one direction or multistructures consisting of different dimensions, it is better to make an asymptotic study which yields models of lower dimension that can be handled more easily. In the same spirit, let us mention that long time integration of equations generally needs the knowledge of the solution and requires a study of asymptotic behaviour of the solution for large times. This means in the modern language, to find out the attractor and the inertial manifold. Based on this, one can suggest the so-called nonlinear Galerkin approximation in which only significant Fourier modes representing the solution are taken into account. The trouble is that there are still too many of them. Here is where wavelet basis may be of immense help. The idea is to group Fourier modes and seek a new representation of the velocity field in

terms of wavelet basis. The research in this line seems to be full of promise.

Linear hyperbolic problems on unbounded domains arise in models of scattering problems. How are we going to discretize such domains? One idea to overcome this difficulty is to look for a suitable boundary formulation of the problem. If scattering frequencies are to be calculated we have to worry about diffraction, grazing, etc. Direct computations, are quite difficult. One can think of using the ansatz of linear geometrical optics which reduces the problem to a set of ODE's and a simple transport PDE. In this context, let us also mention the problem of localization of a shock. This is a difficult problem because the shock is driven by the flow behind it which is enormously complicated. Shock dynamics tries to isolate those mechanisms which are mainly responsible for the shock movements and cooks up a simplified model. Once again, geometrical optics techniques are in forefront and are found very useful. A wide variety of singular perturbation techniques are also available to simplify a given model before starting computations on it.

Another direction in which progress has been made is the acceleration of various algorithms. Parallel computations serve this purpose apart from specific algorithms like Fast Fourier Transform (FFT) and Fast Wavelet Transform (FWT).

3.6 Conclusion

In this brief sketch, we have merely touched upon several aspects of computations which are more or less specific in nature and discussed their connections with various qualitative properties of the solution. Thus merely proving its existence is not sufficient, indeed it is only a first step which provided a framework through which further analysis should be pursued. Needless to mention that many more points are left out in our discussion.

As far as we see, each and every topic discussed above needs attention. However, the following areas may be developed on priority basis in the near future: (i) Kinetic schemes for fluid equations and kinetic-fluid coupling, (ii) Domain decomposition and parallel computing, (iii) Finite volume method, (iv) regularity and asymptotic behaviour in PDE's (v) Turbulence modelling, (vi) Attractors, NGM, wavelets, (vii) Controllability techniques to manage chaos, (viii) Adaptive methods used for various criteria.

Some expertise (but not much) is available in India (e.g. : TIFR (Bangalore), IISc, IIT-Mumbai, IIT-Kanpur, ISI Bangalore, IMSc, etc). In order to strengthen the foundation for a better research in this field, it is necessary to conduct training courses for selected candidates at one of the well-established centres.

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STOCHASTIC PROCESS MODELLING

4.1 Introduction

Representations of natural phenomena evolving in terms of stochastic process models is as old as the theory of probability itself.

These models provide a basis for understanding the phenomena modelled, as well as a tool to predict and control the phenomena.

4.2 Modelling in Biology and Genetics

Stochastic modelling in genetics has a long history. There have been a large number of studies on gene fluctuations over generations. But most of these studies are finite allele models - that is, one assumes that there are only finitely many alleles of the gene. But in the recent past, it has been felt that there are infinitely many (large but potentially infinite in number) alleles of a gene, even though only finitely many of these occur in the population at any given time. This leads to the development of infinite allele models. Stochastic process modelling is also used to model the process of enzyme reaction.

4.3 Queueing

'Queueing systems' is a well-developed area of stochastic process modelling. To effect improvement in our communication systems, there is a wide scope for application of queuing models.

4.4 Perturbations of Dynamical Systems

In studying dynamical systems, one usually has a function arrived at by solving a system of differential equations. The object of study is the nature of orbits of the system, understanding of any periodicities present and stability of the system. If the system is perturbed, then one would like to know its behaviour. This problem is of practical importance because of our limitations on measurements. We cannot measure the state of the system exactly. So, our data reflects a perturbation of the state of the system. These perturbations are usually stochastic in nature. The state of the system, as the perturbation approaches zero, is of considerable importance and involves nice application of stochastic processes. Here one has to model the perturbation itself and study the system.

4.5 Stochastic Recursive Algorithms

There has been renewed interest in stochastic approximation and its variants. This is primarily due to the emergence of new application areas such as adaptive filtering, neural network training, intelligent control, parametric optimization in queueing networks by infinitesimal perturbation analysis, etc., not to mention their use as models of boundedly rational learning in macroeconomics. These often fall outside the purview of classical theory and demand new methods of analysis. They also throw up novel issues such as the problem of asynchronism (i.e., absence of a global clock) in a parallel distributed implementation.

Another class of stochastic algorithms to undergo extensive study has been simulated annealing, particularly popular for large, hard combinatorial problems. It has connections with statistical mechanics and large deviations and has drawn attention of both theoretical physicists and probabilists.

4.6 Large Deviations and Applications

Large deviations theory apart from its traditional standing as a cornerstone of probability, Mathematical Statistics and Statistical Physics is finding an increasing number of applications in diverse areas. Some of these are :

1. queueing network, where it has led to characterizations of important system parameters like equivalent bandwidth and analysis of how congestions/delays come by,
2. fine analysis of stochastic algorithms as in stochastic recursive algorithms' as also in developing the general paradigm of probably approximately correct learning,
3. equilibrium selections in evolutionary games, and
4. various statistical mechanical models.

4.7 Interacting Particle Systems and Related Fields

This has been a thrust area of probability theory for many years and continues to thrive. Some stands of active research are : various statistical mechanical models, growth models, percolation theory, etc. More recent developments include models for self-organized criticality and statistical mechanical analysis of large stochastic neural networks (Boltzmann machines). The continuum case also elicits much interest, particularly specific models like Ginzberg-London or Mckean-Vlasor equations, as well as the general problem of passage to the hydrodynamic limit.

4.8 Filtering and Control Theory

Stochastic filtering theory and stochastic control theory have originated with concrete engineering applications. The theory of stochastic integrals and stochastic differential equations enabled researchers to handle fairly complicated models. This has been an active area of research both from theoretical and application points of view.

4.9 Stochastic Modelling in Physical Sciences

One of the earliest successful uses of stochastic process modelling is in the theory of the diffusion processes by Einstein and Smoluchowski and that led to the understanding of the Brownian motion. The basic problem is that of studying the equations of motion (classical or quantum) in presence of stochastic noise (Langevin equation). If the noise is classical, i.e., modelled by some Brownian functional, then one is looking at (classical) ordinary stochastic differential equations and this subject has developed to a high degree of sophistication. The more recent development is one of (quantum) stochastic differential equations where noise itself is of non-classical (quantum) nature. This is an area of intense research activity at present. While the physicists are phenomena-driven the mathematicians are concept-driven in their research pursuits and hence significant collaborative programmes are desirable.

4.10 Current Status of Research in the Area

A good amount of work has been done in India on stochastic processes and their applications, including characterizations of and inference on stochastic processes. Researchers at the Indian Statistical Institute (Bangalore, Calcutta and Delhi) and at the Pune and Panjab Universities (Departments of Statistics) have contributed to the theoretical aspects of the subject like limiting behaviours and characterizations of stochastic processes, stochastic calculus, theory of diffusions. Contributions to time series analysis, including estimation problems involved therein, have been made by the Departments of Statistics of the Calcutta and Madras Universities. Some researchers in ISI (Delhi) and the Calcutta University have contributed to filtering and prediction problems, while problems of stochastic filtering and control have been pursued in ISI (Delhi) and the Indian Institute of Science (Bangalore).

Besides the institutions mentioned above , the stochastic process modelling in the area of queues has been studied in several universities, in the Indian Institute of Management (IIM), Calcutta and in Indian Statistical Institute (ISI).

Some work related to applications in Genetics and Epidemeology has been carried out at ISI (Calcutta), Pune University and Indian Agricultural Research Institute (IARI), Delhi. Similarly, use of (stochastic) process models for optimal process control in industry has been studied at the Calcutta and Karnatak Universities.

Recent research activities on stochastic process modelling in Physical Sciences is essentially concentrated in the School of Physical Sciences Jawaharlal Nehru University(JNU), New Delhi, S.N. Bose National Centre for Basic Sciences, Calcutta, and in the Central University, Hyderabad, and in ISI, Delhi.

4.11 Areas of Gap in Research

It appears that mathematical models in terms of differential equations involving deterministic state variable(s)-sometimes coupled with deterministic covariates-are being used rather widely at various universities, research laboratories and allied institutions-even to understand, predict and control interaction takes place between researchers on stochastic processes and the number of people interested in applications vision stochastic process modelling has been somewhat small.

An understanding of the modern theory of stochastic processes requires a fair amount of mathematical background. Many researchers who deal in applications of the theory are handicapped on this score. It is felt that researchers working in this area should ensure that they have adequate background in the basics of probability theory, namely, limit theorems, Markow chains in discrete and continuous time and martingales (in discrete time).

Regarding perturbations of dynamical systems, many interesting results have been derived, though a definite theory is yet to be developed. Applications of filtering and prediction theory calling for necessary modifications to problems like signal decoding, image processing and weather prediction have not yet been taken up in our country. This is worth exploring - especially because with the advent of our indigenous satellite data it should be possible to develop and test the models. At the moment, only time-series analysis is the tool used to understand earthquakes of various magnitudes in a given region. A clear and complete analysis of infinite allele models has not yet been attempted nor has stochastic process modelling found adequate uses in bio-technology. Transient solutions of many complicated models needed to represent the dynamics of real life systems do not admit of elegant solutions and require further studies. There are only a few studies on turbulence, but the area seems wide open.

The following are some of the suggestions to promote the study and applications of Stochastic Process Modelling :

(a) Organise seminars /workshops involving a select group of workers from universities/ national laboratories/ research institutes who pursue modelling and some selected workers in stochastic processes to work out a mechanism for strong interaction between the two groups.

(b) Organise workshops on specialised topics, preferably at places where a core group on the topic exists.

(c) Organise orientation courses at some of the Institutions having a group of prospective end users to make their scientists appreciate the use of stochastic process modelling.

(d) Encourage researchers on parameter estimation in stochastic processes and tests for validation of selected models in order to promote the use of stochastic process models.

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DETERMINISTIC CONTROL THEORY

5.1 Introduction

Control theory is concerned with modifying the behaviour of dynamical systems to achieve desired performance, usually by applying appropriate inputs to the system. Thus, the discipline is divided into two sub-branches, namely: deterministic control theory, and stochastic control theory. In this brief write-up, a description is given of some of

the currently popular areas in deterministic control theory that have substantial mathematical content. Topics in stochastic control theory are covered under Stochastic Process Modelling.

Within deterministic control theory, one can make a further subdivision into linear and nonlinear control systems. The division is not very "sharp", however, because any smooth nonlinear system can be locally approximated as a linear system, and controllers designed on the basis of linear control theory will perform satisfactorily in a suitably restricted regime. However, the branches of control theory tend to be somewhat different.

Owing to advances in control theory during the past thirty years or so, the mathematical problems underlying typical engineering applications of control theory are well-understood and well-formulated. That does not mean they are solved! Usually control theory has its greater impact in those engineering situations where high-performance is required, e.g., fighter airplanes, design high-precision and high-speed robotics, altitude control of spacecraft, and so on. There are other control engineering applications in which time scales are rather long, and nonavailability of accurate models is the overriding factor rather than high-performance requirements. Examples include nuclear reactors, paper mills, boilers for thermal power plants and so on. Past experience indicates that mathematical methods are less well-suited to such applications.

In control theory, an important step is that of translating engineering requirements regarding the desired performance of the system, the nature and extent of the uncertainty in the data, etc., into a suitable mathematical framework, which are not unique. Thus, it may be possible to view the same underlying engineering problem in control system design from the standpoint of several branches of mathematics, such as functional analysis, algebraic geometry, differential geometry, and so on. Each discipline might bear a different set of insights into the problem formulation and its solution, and all need to be encouraged.

But it is important to note that, whichever branch of mathematics is used to model the engineering problem, the resulting mathematical questions are often very deep, and at the same, decidely nonstandard.

We now turn to some specific topics of research, which are relevant at the present time and are likely to remain so for the next five years at least. Of course, the list below should be revised from time to time so as to reflect changes in the engineering branch of control theory.

5.2 Linear Optimal Control Theory

Since 1980, linear control theory has been dominated by the so-called H_∞- optimal control theory, and its later variant, h_1-optimal control theory. Both types of optimal control problems can be captured in a common mathematical framework, which is now described. Let β be a commutative Banach algebra with identity. (In the case of H_∞-optimal control, β is the Hardy space H_∞ consisting of functions that are analytic on the open unit disc and essentially bounded on the closed unit disc. In the case of L_1 - optimal control theory, β is the set of absolutely summable sequences, or equivalently the set of analytic functions on the open unit disc whose power series are absolutely convergent on the closed unit disc.) Let A, B, C be given matrices of compatible dimensions, whose entries belong to β. The problem is to choose a matrix R, whose entries also belong to β, so as to minimize J(R) :=||A-BRC||.

The geometric nature of the problem is easier to see if A, B, C are scalars rather than matrices. In this case R is also a scalar, and one can combine B and C into a single element because multiplication is commutative. We use lower case letters to emphasize the fact that we are dealing with the scalar case, and use only b, instead of both b and c. In this case the problem becomes one of minimizing. J(r) := lla-b ll as r varies over B. Geometrically the problem is that of finding closest element to 4a in the principal ideal in B generated by 4b. lf, clb= H_∞ this problem reduces to the classical Nevallina-Pick interpolation problem. In the

matrix case, if β= H_∞ the problem can be similarly interpreted as a minimum-norm interpolation problem provided the matrix β is 'fat' (has at least as many rows as columns), and the matrix C is 'tall' (has at least as many rows as columns). However, the general case is much deeper. The case where β=l_1 is not yet wellunderstood, because of the paucity of norm-preserving transformation.

The investigations into the above problem can be divided into two broad categories, namely: geometric and computational aspects. The 'geometric' aspect pertains to the way in which one thinks about the problem, and is useful for the derivation of some qualitative results, such as the existence and uniqueness of a minimizing R, and some insights into its nature. The 'computational' aspect deals with efficient computational algorithms for determining an optimal R. Both areas are quite wide open at the moment.

At the moment, there are only two book-length treatments of this problem [1,5]; both are written in a manner that is quite accessible to mathematicians.

5.3 Sampled Data Control Systems

This is a somewhat more recent area than the previous one and as such there is much more scope for mathematicians to make a contribution. The actual research problems in this area appear superficially to be similar to those discussed above, but with some important differences. First of all, because of sampling, even if the underlying engineering system is single-input, single-output the resulting set B is usually very complicated. At the bare minimum, it becomes a Hilbert space L_2 ([0,T]/H), the space of square-integrable functions mapping a finite interval [O, T] *into an infinite-dimensional* Hibert space H. Moreover, the natural definition of the 'multiplication' operation in this case makes it noncommunicative always. There are no books as yet in this field.

5.4 Differential Geometric Control Theory

During the past decade or so, differential geometry has played a very successful role in analying various aspects of nonlinear control systems. This approach has been able to tackle several diverse problems, such as controllability, observability, and feedback linearization. The last application is particularly notable. The object of study is a nonlinear system in which the control inputs appear in an affine-fashion, of the form $x=f(x)+\sum_{l-1}^{m} g(x)u_i$ where $f,g_1...g_m$. are vector fields. The objective is to apply a *feedback control* of the form u=m(x)+N(x)v, where m(x) is a smooth vector, N(x) is a smooth invertible matrix, and v is the new vector of control inputs, together with a *coordinate change* of the form x=T(z), where T is a local diffeomorphism, in such a way that the resulting system is described by the set of linear vector equations z = Az+Bv. Necessary and sufficient conditions for the existence of a solution were derived more than a decade ago, and book-or chapter-length description of the results can be found in [3,4,6]. Moreover, the theory has been applied quite successfully in several areas, for example, in robotics, (both to rigid robots as well as robots with elastic joints), helicopters, and certain types of electric motors. Some of these applications are described in [6].

However, many interesting problems still remain. Most of the successes to date deal with the *local* theory of non-linear systems. As is clear from many other situations (for example, from examples in theoretical physics), global situations are constrained by topological and geometric obstructions. Obviously, global versions of the above problems pose an important challenge, because they would need to take into account the topological features of the underlying state manifold. The problems described above are predicated other assumption that the feedback control law as well as be *smooth*. This in turn requires that certain distributions be nonsingular-an assumption that is not satisfied in many natural applications (e.g., robots containing flexible links). Thus the question arises as to what can be done in such a case. Instead of feedback linearization, one may ask more generally the question of

characterizing systems that can be made simple in other ways. For example, a challenging problem is that of finding conditions under which a system is feedback equivalent to one whose dynamics is described by vector fields which generate a nilpotent Lie algebra. In particular, one has the following problem in geometry: Given a singular distribution on a manifold, decide whether it admits a basis consisting of vector fields generating a nilport Lie algebra.

5.5 Control of Hybrid Systems

Hybrid systems are consisting of processes given by both continuous and discrete events. Such systems abound today with the availability of technologies that network. Process controllers with various communication and switching devices. The discrete state of some components of the system will govern the evolution of the continuous states of other components and conversely, the continuous state will govern the transition of discrete states. Any mathematical fomalism used to model and analyze the behaviour of such hybrid systems has to seamlessly blend the algebraic and geometric tools of nonlinear systems methodology with the methologies, of discrete event systems (viz. automata theory, formal languages, modal logics). The jury is still out on the choice of formalisms with the three front runners being suitable enhancements of the methodologies of discrete event systems motioned above. Hybrid systems represent a frontier research area for applied mathematicians since the problems are driven by real systems for which engineers would really like to have robust theoretical constructs to base their control laws on. An edited collection of papers on hybrid systems [7] is perhaps the most accessible source for an introduction to this new research topic in control theory.

While there is no shortage of challenging control problems in our country (e.g., the various missiles, the various spacecrafts etc.), the general level of expertise in our academic institutions leaves a lot to be desired. To raise the level, it is suggested that a series of training programme are designed in a few areas discussed above.

Some institutions where expertise in the type of mathematics used in control theory is available are :

1. TIFR, Mumbai
2. SPIC Mathematical Institute, Chennai.
3. Central University, Hyderbad.
4. IIT, Mumbai.
5. ISI, Calcutta.
6. Centre for Artificial Intelligence & Robotics (CAIR), Bangalore.
7. University of Bombay, Mumbai.

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EXPLORATORY DATA ANALYSIS

6.1 Introduction

Exploratory Data Analysis (EDA) refers to a collection of statistical methods that try to explore important and interesting features in the data and utilize them in the process of empirical model building as well as for making useful inference and

predictions. As opposed to more traditional techniques, where typically the analysis focuses on an explicitly laid down mathematical model, EDA techniques dig around the data looking for informations and incorporate them in the model that evolves gradually through such an empirical analysis.

Many of the procedures that currently form the toolkit of EDA have been developed by scientists and engineers in addition to statisticians, and they have great relevance to various kinds of scientific investigations and inference. Some of the EDA techniques that have received extensive attention from researchers as well as users and have achieved prime importance as fundamental data analytic tools are listed below.

1. Diagnostic studies for judging validity of assumptions or suspicions and detecting peculiarities in the data such as outlying observations.

2. Empirical model building strategies such as various types of nonparametric smoothing and function estimation techniques, model selection using data-based adaptive criteria, tree structured methods, cross validation and evaluation of empirically developed models, etc.

3. Resampling procedures such as the jack-knife, the bootstrap, etc., for quantifying and rectifying (whenever possible) uncertainties and biases in statistical procedures based on uncertainties inherent in the data itself.

4. Exploration of distribution and Euclidean geometries of multivariate data clouds and model-free multivariate inference.

   There are many potential applications of EDA in scientific investigations that have direct impact on mankind and the society. Besides, EDA is useful in many research studies that involve challenging problems of understanding complex natural processes. A few important areas where EDA can have an effective role are mentioned below.

a. Modelling and analysis of seismic data for getting insights into the geophysical processes behind earthquakes and developing possible procedures to make forecasts based on patterns in observed events.

b. Modelling and analysis of meteorological data to understand periodic patterns in natural phenomena like rainfall, monsoon as well as to develop forecasting systems for disasters such as storms and cyclones.

c. Processing of images (e.g., those recorded by satellites) for identifying patterns and peculiarities that are of crucial importance.

d. Analysis of high dimensional noisy large data sets resulting from large scale investigations (e.g., biomedical studies, industrial experiments, etc.) for figuring out useful low dimensional structures in them.

Several EDA techniques were originally proposed on the basis of *adhoc* logic, and they have been found to be quite successful empirically. There are many unsolved theoretical problems concerning the performance of those EDA procedures when judged from an objective mathematical ground. Besides, many of the EDA procedures and ideas need to be translated into user-friendly algorithms so as to make them implementable in practice. One would hope that EDA will attract the attention of researchers as a stimulating multidisciplinary area of research with tremendous potential of real applications in future.

6.2 Status of current research in India

Only a small amount of work has been done in India on EDA over the last few years. Some statisticians at Indian Statistical

Institute, Calcutta and Calcutta University and Pune University are actively involved in research projects that focus on specific aspects of EDA. These include various forms of exploratory regression analysis (e.g., model and variable selection, nonparametric smoothing and function estimation, tree structured regression, regression diagnostics and robustness studies, resampling techniques and geometry of multivariate data clouds. On the other hand, scientists at various research centres and labs in India (e.g. CSIR Centre of Mathematical Modelling and Computer Simulations, Bangalore, Indian Institute of Tropical Meteorology and National Chemical Laboratory, Pune, National Center for Medium Range Weather Forecasting, New Delhi; National Geophysical Research Institute, Hyderabad, etc.) are involved in empirical modelling and related analysis of different types of meteorological and seimic or other types of geophysical data. Often they use innovative EDA techniques as apparent from their articles and short communications published in journals like Proceedings (*Earth & Planetary Sciences) of Indian Academy of Sciences, Current Science, etc.*, though the authors of those articles hardly describe (or perceive) them as EDA techniques !!

6.3 Areas of gap in research

It is clearly felt that the need for methodological and theoretical research on EDA has not been fully appreciated yet by a large fraction of our researchers and teachers. Also, there seems to exist a significant communication gap between statisticians and other scientists who are using EDA techniques appropriate for their specific scientific problems. The absence of statisticians and mathematicians in the whole spectrum of many scientific investigations and experiments appears to be a barrier obstructing clear identifications and development of useful EDA techniques and their refinements that can be achieved through theoretical research, in a coubarative environment.

6.4 Measures to promote research

There are some major steps that should be taken to improve our overall scientific strength in EDA. They are briefly discussed below.

i. There must be some serious initiative to create more interest among teachers and researchers about EDA. This can be done through organized lectures by experts as well as workshops organized at universities and institutes. Also, since most of the EDA techniques are computer intensive, efforts should be made in creating appropriate computing environment at unversities and institutes so that research and training on EDA can be sustained at a reasonable level.

ii. The communication gap between statisticians and other scientists sharing a common interest in EDA must be eliminated and replaced by active interaction and collaboratiton. There is need for regular exchange of views between statisticians and other scientists with some focus on specific scientific problems (e.g., monsoon, earthquakes, cyclones, satellite images, etc.,) This can be achieved by promoting collaborative research projects between statisticians and other scientists as well as small conferences with special theme topics, where statisticians and other scientists will participate jointly.

iii. Statisticians and mathematicians should become involved in the entire spectrum of real scientific studies and experiments taken up at various national research labs and have the opportunity to interact with and give their inputs to the scientists there.

EDA is a multidisciplinary technology that has a number of challenging theoretical and methodological research problems. Associated with it research and training on EDA can be promoted and fruitfully sustained only if there exists active interaction between statisticians and other scientists having well defined scientific problems in their focus.

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DYNAMICAL SYSTEMS

7.1 Introduction

The subject of dynamical systems (DS) is so vast and varied that it is impossible to address all the problems and questions treated in it. We thus propose to confine ourselves to a certain important corner of the subject and discuss some significant issues. The discussion centres around questions arising in various applied fields such as classical mechanics (CM), fluid mechanics (FM), economics, biology, electrical circuits, etc.

We start with a smooth manifold *M* which represents the state space of the system. By a DS, we mean either a smooth diffeomorphism $f : M \to M$ or a vector field (vf) *F* on *M*. The evolution is then modelled by the iterates $\{f^n(x);\ n \in \mathbb{Z}\}$ or the flow $t \to X(t,x)$ generated by *F* on *M*, i.e. $dX/dt = F(X),\ X(O, x) = x$. In sequel, we will alternate between these two depending on the simplicity of the presentation. As an immediate example, one can take $M=D$ where *D* is a smooth, bounded open set in $\mathbb{R}^d$ and *F* is a vf on *D* which is tangent to the boundary of D. The central problem of DS can be easily phrased as follow: what is the behaviour of flows as $t \to \infty$ or $n \to \infty$? To bring out the difference in qualitative properties, one must distinguish between the following cases: *M* finite or infinite dimensional, *f* invertible or not, dissipative or conservative system. In the case of PDFs, we will have infinite dimensional manifold *M*.

7.2 Well-definedness of DS

To enable one to discuss the properties of dynamics, it is necessary that they be uniquely defined for all times. Existence proof is usually carried out in two steps: first one obtains local solution which can blow up in finite time. In order to prevent this, we try to establish *a priori* estimates. If we succeed, then local

solution becomes global. For example, unique local solutions are assured if *F* is of class C^1. They become global if *F* is Lipschitz or *M* is compact. There are many examples of nonlinear PDEs where existence of suitable global solution is not known. One also comes across another kind of difficulty. In the important case of Navier-Stokes equation (NSE) in three dimensions one can prove the existence of a suitable global weak solution, but it is not known whether it is unique or not. This question is related to the regularity of weak solutions, an issue which is not satisfactorily resolved in the case of many important nonlinear PDEs.

Let us point out another important direction which has received less attention; it is the case of singular vf. A classical example is the *n*-body problem. Another one arises if we wish to define the dynamics of dye carried by fluid flow. One idea[1] is to exploit the variational chacterization of the trajectories. Another is to approximate singular vf by smooth vfs. In Diperna and Lions[2], this has been successfully carried out with the notion of *almost everywhere* (ae) flows, i.e., the flow is defined omitting a null set of initial conditions (IC). Perhaps, this is the only work which follows the point of view of Poincare and throws away naturally an exceptional set of non-representative’ trajectories. Many qualitative aspects discussed below are yet to be investigated in the case of ae flows.

Nonlinear hyperbolic equations give birth to another type of singular vsf. Because of the appearance of shocks and other sigularities in these equations, the corresponding vfs are not well understood. Nevertheless, let us mention that there is another concept of solution, named after Fillipov, associated with singular vfs. It is defined pointwise and is useful in hyperbolic equations[3].

Because of these difficulties, in the sequel, we work with a compact manifold *M* without boundary and smooth vf *F.* There are not many works which incorporate the effects of boundary and the behaviour at infinity.[4]

7.3 The main problem

The goal is to draw the *phase portrait* of the flow; especially, we are interested in its behaviour as $t \to \infty$. This fascinating subject has attracted the attention of several great mathematicians like Poincaré who investigated the stability of the solar system. Fluid turbulence is another model phenomenon. This problem remains essentially unsolved because of the rich and varied behaviour of DS. In olden days, resolution of ODEs was achieved by obtaining smooth invariants. This yielded positive results in cases now known as *completely integrable systems*. A major result in this area is Liouville's Theorem[5]. When it was realized that nonintegrable systems are the order of the day, other approaches were sought. There are five major methods.

(A) Geometric and topological point of view initiated by Poincare,

(B) Statistical approach originating in the works of Boltzmann and Maxwell,

(C) Algebraic formulation of Koopman for which we refer to Arnold and Avez[6],

(D) Numerical integration, and

(E) Nonlinear functional analytical approach.

(A), though old, is dominant even today. After the invention of computers, method (D) has become powerful and enabled scientists to get insight into the behaviour of DS. In the context of fluid turbulence, it has been felt that (A) is helpful in understanding its onset while moderately excited regimes require new tools provided by (B). In fully developed turbulence, no method is found suitable except perhaps (D). Method (E) is relatively new and is proving very powerful in the analysis at large. It is especially very effective in the case of Hamiltonian systems when combined with other methods. The method invokes variational properties of the

trajectories. In the sequel, we do not discuss this fast-growing approach except to give some important references. [1,7-11] In the important field of infinite-dimensional DS, the techniques of nonlinear functional analysis are proving to be very useful whereas the other methods have their limitations.

7.4 Part A: Geometrical approach in finite-dimensional case

The classification problem: Whatever be the method of attack, the canonical programme towards the understanding of the dynamics seems to be the following:

Step (1) Identify two dynamics with the same behaviours,i.e., define an equivalence relation in *V*, the set of vfs on *M*. Present the phase diagram inside each equivalence class. Also obtain a canonical form of the vf in each equivalence class. Define a vf to be *stable* if it has neighbourhood of equivalents in *V*.

Step (2) Prove that stable vfs are dense in *V*.

Step (3) Characterize the stable classes in simple terms. Classify them in terms of invariants (algebraic, numerical, etc.).

Step (4) Classify the unstable classes of codimension 1.2....

Step (5) Study the bifurcation at unstable classes and the nature of the unfolding.

Let us explain the above programme. Among the many equivalence relations used depending on the context, we single out *topological conjugacy* which is defined in terms of homeomorphism mapping the orbits onto themselves. Stability w.r.t. this equivalence is known as *structural stability*. Obviously, this is an important concept if our model is to represent the reality. If our model is not stable then we must be able to choose a perturbation.

This is Step (2) which has remained a dream in the theory of DS. Our equivalence relation should be fine enough to distinguish things that are qualitatively different but sufficiently coarse to prove Step (2). Secondly, weaker the topology on V the more chance we have to establish Step (2). It is a common practice to use C^k-type topology on *V* but now there is a weaker topology which is found more natural in view of Diperna and Lions.[2]

The unstable elements, hopefully, will form submanifolds of finite codimension which are to be classified. The significance is that any r-parameter family of vfs inside V will generically cross only those submanifolds of condimension $\leq r$. At this crossing, one can expect bifurcation and new qualitative behaviour.

Physical and numerical experiments exhibit various 'stable' phenomena which are not stable structurally. In practice, one may thus run into difficulties with the concept of structural stability because we often hāve to deal with a restricted class of vfs in which case we will not accept arbitrary small perturbations inside *V*. Stability is thus to be understood in a broad sense.

In the classification problem it is always advocated to ignore vfs and orbits which are not representative. The genericity of a phenomenon is therefore usually evaluated from its validity on a large set in a topological sense (eg, Baire set) or in a measure sense. It is true that there is no canonical measure on V, but, if we restrict our attention to vfs depending on a set of parameters we can take measure induced from the space of parameters.

In these questions of classification, smoothness of vfs plays an essential role. In particular, one can see in the literature different behaviours depending on whether a vf is C^1,.. $C.^2$,... C^n, C^w. Another related question of importance is the following : if a vf is smooth and it is equivalent to another, can one choose the equivalence map to be smooth? These questions are not answered in a general way because of the presence of resonances.[12]

Local classification: The classification problem is solved at each point of M in a satisfactory manner. [13,14] We are, of course, interested in a local description which depends on the interaction between dynamics and the geometry of the manifold. The interest in the local description is that it brings out certain fundamental concepts which are then generalized to attack the global problem. One such notion is that of *equilibrium points*, i.e., points at which vf *F* vanishes. If it does not then the flow is equivalent to a tabular flow. In the local study it is natural to linearize the flow at an equilibrium point $x^\star$. Stability of orbits then depends on the distribution of eigenvalues $\{\lambda_i\}$ of the linearized operator. One realizes the importance of the notion of hyperbolicity (i.e. Re $\lambda_i \neq o$ $\forall$= Flow around $x^\star$ is then governed by *stable* and *unstable manifolds:* W^s $(x^\star)$ and W^u $(x^\star)$. $x^\star$ is a *saddle* if dim $W^s \neq 0 \neq$ dim W^u. Hyperbolicity is a generic property. A flow near its hyperbolic equilibrium point is equivalent to its linearization. Linear hyperbolic flows are essentially characterized by their indices (=dim$W^u(x^\star)$). At nonhyperbolic points, one has also to deal with the *centre manifold*.

Gradient vector fields: Here we refer to those systems where the vf F=- ∇g for some functions g called potential. In this category, it is natural to consider perturbations of g rather than those of *F*. The resulting stability concept is different from structural stability. A function is stable if has a finite number of critical points, each nondegenerate and having distinct critical values. Thorm[15] has characterized the unstable function classes of codimension ≤4 in terms of singularities. They are just the elementary catastrophes! He also assumes that one can pass from the bifurcation of gradient dynamical systems to the unfolding of their potential functions in studying catastrophes. This is not entirely correct. The unfolding of gradient dynamical systems can be of higher dimension than the unfolding of their potentials.[16]

Two-dimensional flows: The landmark results characterizing two-dimensional flows are theorems of Andranov-Pontrayagin, Poncaré-Bendison and Peixoto. One of the new generic phenomena exhibited is that of (hyperbolic) *perioudic orbits,* i.e., the w and limit sets can be only equilibrium or periodic orbits. Further, saddle connections are not allowed. For these results, see Palis and de Melo.[13] Though there are vfs which are not stable (e.g., a quasi-periodic motion on a 2-torus where the frequencies are independent over Q; the orbit is then dense), the stable ones are dense. What about the characterization of stable vfs in terms of invariants and the classification of unstable ones? It is not clear whether, these questions have been completely answered. See, however, Hale and Kocak.[17]

If instead, of two-dimensional flows, we consider diffeomorphisms on one-dimensional manifolds. say S^1, then *rotation number* allows us to classify the maps.[12]

Stable systems: In trying to generalize the above to general flows, one runs into enormous difficulties. This can be vaguely explained as follows : 2d flows correspond to maps in 1d and there is a natural ordering in $\mathbb{R}$ which can be exploited. In dimensions ≥ 3, there are other stable phenomena which will be described in this paragraph. Even after adding these, the density (Step 2) is not true. The classification programme is a real challenge posed by nature to mathematicians. Efforts are now on to collect various possible bahaviours and it is hoped that the density will be proved some day in future.

Smale[18] was one of the first to group various known examples and generalize them to higher dimensions. It is known for a long time that recurrence properties play an essential role in the study of asymptotic behaviour of DS. Thus, *non-wandering* set Ω which includes equilibria and periodic orbits was introduced. Next, Morse-Smale systems were introduced where Ω consists of

finitely many equilibria and periodic points (all hyperbolic). Next, the condition of 'no-saddle connection' is replaced by the *transversality* of stable and unstable manifolds of elements of Ω. We emphasize that the intersections of the stable and the unstable manifolds have to be preserved by any topological equivalence. It is therefore natural to require that these intersetions be transversal since this will guarantee that they persist under small perturbations.

Even though M-S systems are stable,[13] they are far from being dense. World is not as simple as M-S systems. Indeed, a much richer structure was noticed by Poincaré himself at a *transverse homoclinic orbit.* In particular, the system was sensitive to initial conditions (SIC) near it. Such systems are called *chaotic*. Smale noticed three main mechanisms responsible for this effect: contraction, expansion and folding of state space volume by trajectories. Using these, he constructed his *horeseshoe* where in contrast to M-S systems, there are infinitely many periodic saddle points coexisting. Moreover horseshoe is stable. It is worth remaking that the set of IC attracted in a horseshoe has measure zero if the system is of class C^2. However, there are C^1 horseshoe examples of positive measure.

Of course, there are other types of stable systems, e.g. Anasov systems where Ω is the entire manifold. Examples include the geodesic flows on a manifold with negative curvature.

Generalizing these objects, Smale introduced the notion of (uniformly) *hyperbolic sets* as associated with flows/maps. They are compact invariant sets at every point of which there are contracting and expanding directions. Such sets are stable under pertubations of the map. Hypebolic systems are the ones for which Ω is a hyperbolic set. A system is said to satisfy *Axiom-A* if it is hypergolic and set of periodic points is dense in Ω (which is true generically[19]). For such systems, Smale obtained the following satisfying picture[18]: there are finite number of *attractors* (compact invariant sets whose basin of attraction contains a neighbourhood of it). Basins put together cover a dence open subset of the manifold. Each attractor is transitive (it has a dence orbit) and is contained in Ω . Further, attractors which are not just fixed or periodic sink exhibit SIC. They are called *strange attractors*. Thus, Axiom -A systems can be decomposed into Anasov pieces assembled together somewhat like M-S case! This result can be viewed as a nonlinear analongue of the decomposition of the space in terms of generalized eigenvectors of a matrix.

The culmination of this circle of ideas is the following remarkable result of Mañé 20 : a diffeomorphism is C^1 structurally stable if it satisfies Axiam-A and all stable unstable manifolds are transversal. Thus, we have a grand picture of structurally stable diffeomorphisms and their dynamics. The role of hyperbolicity in this cannot be overemphasized. The corresponding question for flows remains essentially unsolved.

From this analysis, it is clear that one must have efficient algorithms to find equilibria, periodic points, their stable and unstable manifolds, homo-and heterro-clinic orbits and criteria, to test their transveral intersection. Some tools are dynamical zeta function[18,21] and Melnikov technique.[22] Many more are required.

Unstable systems: Having obtained a nice picture of stable systems, we might ask whether they are dense. It is known for a long time that they are not. Structural stability is thus of more limited significance than anticipated. The world of dynamics is very rich and fascinating. The classfication of unstable systems and the resulting bifurcation is a problem that remains essentially unsolved. Attempts are being made to understand them by looking through the boundaries of stable ones. For instance, one may consider one parameter family of systems $\{F_\mu\}_{\mu\varepsilon IR}$ such that F_μ is stable for μ<0 and μ<0 and F_0 is not stable. One expects different qualitative behaviour as $\mu \to 0$. To understand the situation, the concept of *attractors* is useful. At the bifurcation point μ=0 there is a change

in the topology of the attractor. Multiparameter bifurcations are poorly understood.

In literature[23,24] one can observe a long list of unstable situations. On one hand, one considers the cases where stable and unstable manifolds are not transversal or Ω loses hyperbolicity. On the other hand, there are scenarios obtained by Ruelle-Takens, Feigenbaum, Manneville-Pomeau, Kenon, etc. In each of these scenarios, not only a description of the attractors involved is presented but also unfolding of them (i.e. the route which yields them) are also given. In Ruelle-Takens scenario, it is shown how a stationary point becomes unstable and gives rise to a periodic orbit via Hopf birurcation. A 2-torus then appears through another birurcation. If another instability occurs then typically a stage attractor appears instead of 3-torus. This is in sharp contrast to the picture projected by Landau and Hopf in the content of the onset of turbulence. Period doubling cascades occur as unfoldings of Feigenbaum attractor. What is surprising is that all these bifurcations are often really seen to follow each other and to converge asymptotically on a geometric sequence. In other words, in the space of maps of the interval there seems to exist a ‘Feigenbaum manifold’ of codimension 1 which is geometric limit of bifurcation manifolds corresponding to period doubling. In the intermittency route proposed by Manneville-Pomeau, the system oscillates in a regular fashion and is stable under small peturbations upto a critical value of the parameter appearing in the system. Beyond this critical value, the system exhibits abnormal fluctuations from time to time.

These systems are not structurally stable but are stable in some restricted sense. The big question is whether the union of these along with Axiom-A systems forms a dense subset in *V*. Are more phenomena to be included? There are several conjectures.

Nowadays, attention is focused on non-hyperbolic systems. Homoclinic bifurcation then becomes importnat and this can be obtained through homoclinic tangencies, for instance. The work of Newhouse [25] is pioneering in this context. Another important breakthrough is achieved to understand the Henon map.[26] There is also a progress towards mathematical basis to explain Feigenbaum cascades and universality. [23] However, a lot remains to be done. Lorenz attractor is poorly understood,[27] and there are many conjectures.[24] Intensive research is on to prove them. Only time will tell if they are a success or a failure.

Even though several mechanisms producing instabilities are known, it is not clear whether a given system undergoes bifurcations when the parameters cross through critical values. It is an open problem whether a given model exhibits SIC. Bifurcations in the presence of symmetry is another vast area which we have not touched.[28,29]

In dynamical problems, the following questions are usually raised and the answers are hard to obtain: how the trajectories are attracted towards the attractors, the rate of attraction, the nature of motion of the attractor, topology and geometry of attractors and their basins, etc. These things keep changing as parameters are varied and at the biruction point, one expects drastic changes. Mandelbrot[30] has been advocating *fractal geometry* to study attractors.

Another question that may be posed is the following: what happens to the dynamics under stochastic perturbations? In other words, we replace ODEs by stochastic differential equations and ask similar questions. There is also intense activity to generalize the above to delay differential equations.[31]

7.5 Part B: Statistical approach

As in Part A, we concentrate here on finite-dimensional DS. The geometric approach presented in Part A has enabled one to attack problems with a few degrees of freedom and thereby explain the

onset of turbulence. There are difficulties with large degrees of freedom. For instance, fully developed turbulence is out of reach for the moment. However, there are physical models where only a moderate number of modes are excited, e.g., flame propagation and combustion problems. To understand such chaotic systems we require new tools provided by *ergodic theory such as dimensions, entropy and Lyapunov characteristic exponents.* Dimension represents the number of excited modes. The inverse of entropy quantifies the time up to which the state can be predicted with precision $0(\varepsilon)$ if IC is specified with tolerance ε. Characteristic exponents describe sensitivity to IC (SIC). In this approach. one deals with a measure μ invariant under the dynamics which replaces invariant sets of Part A. It is then natural ot generalize hyperbolicity as follows: μ is *hyperbolic* if μ-almost all points are hyperbolic, i.e. characteristic exponents are non-zero μ ae. The goal of this approach is to prove that these quantities exit, discover the relations between them and use them to extract qualitative bahaviour of DS. The theory is quite developed[5,32-34] especially w.r.t. moniliform hyperbolic attractors, A spectacular application of these tools will be pointed out in Part C. One of the major problems for the future is to know how the descriptions given in Parts A and B change when one takes, say thermodynamic limit, i.e., when the number of degrees of freedom goes to infinity in a certain sense. Does it give a reasonable picture of continuous systems? What properties are lost in this passage? Reversibility? Another major difficulty is that there are too many measures invariant under the dynamics (e.g., Henon map). Which one is most relevant? In this context, SRB measures were introduced but proving their existence is a hard mathematical problem. Roughly, these measures represent the time spent by the orbits near the attractor. For Axiom-A systems, such measures exist and this is the content of the Bowen-Ruelle Theorem.

Most of the mathematical work in this approach has been restricted to either completely integrable or completely chaotic (ergodic) systems. Little work has been done in the case of intermediate systems which form the bulk of what is encountered in practice.

7.6 Part C: Infinite-dimensional systems

There is a large and growing industry to extend whatever we have said about finite dimensional systems in Parts A and B to the infinite-dimensional case, in particular, to the systems of nonlinear PDEs.[35-38] One of the impressive results is that the NS equation in two dimensions has a finite-dimensional attractor. The same is also true in three dimensional case provided we assume the existence of a unique solution. The estimate on the number of degrees of freedom predicted by Kolmogorov theory of turbulence is thus recovered. We are not going to dwell on how this result is proved. We merely point out two radically new aspects in infinite dimension of which little is known.

(i) We have been discussing about what are known as *temporal chaos*. In PDEs one can also have *spatial chaos*. Examples include flow past a sphere where chaos develops in the wake region. Similar situations arise in turbulent jests and plumes.

(ii) There is a possible occurrence of singularities in space. For instance, it has been conjecturned that curl of the fluid velocity (obeying incompressible Euler equation) can become infinite in some parts of $\mathbb{R}^3$ at finite time. It has been proved that this set has to be small. But one does not know whether this is empty or not. It is also conjectured that such a set is fractal. There is some numerical evidence supporting this.

Another example is the appearance of shock waves. In these cases. the space, in which dynamics takes place is to be so chosen as to include these singularities. Unfortunately, one then risks to lose the uniqueness of solution if the nature of the singularities is not

properly understood. In many practical problems, this difficulty exist.

7.7 Part D: Numerical approach

The biggest question is how to do stable numerical computations in nonlinear systems which exhibit instabilities, bifurcation and SIC, and even if we can do, is there a basis to rely on them? In the hyperbolic case, there is *shadowing lemma.*[19] Non-hyperbolic situations should be looked into. If the system is governed by PDE and the dimension of the attractor is large (which is usually the case) then the power of present-day computers does not allow us to integrate the equations for long times. That is where the insight gained out of the theories developed in Part A, B and C is going to be very useful. Long-time integration demands a good approximation of the attractor. When it has a complicated structure, this is not going to be easy. Thus was born the concept of *inertial manifold* which contains the attractor, is reasonably smooth and attracts orbits in an exponential way.[36] If N is the dimension of the attractor, it does not mean that the first N Fourier modes are sufficient to describe the motion. Because to the complicated geometry, the choice of the modes are subtle. Let us briefly indicate the ideas: split the unknown u into large and small 'eddies': $u=y+z$. Inertial manifolds are sought in the form $z= \phi(y)$. Next, the idea is to project our equations into this manifold. These projections can be done in various set-ups: finite-difference method (FDM), finite-element method (FEM), spectral method (SM) and wavelet method. In literature, one sees at least two ways of achieving inertial projections: nonlinear Galerkin method[40] using SM and incremental unknown method [41] using FDM. These are worked out and tested in only some examples. Much more remains to be done; for instance, one can employ wavelets here.

Numerical computations also offer some first-hand clues on the possible behaviour of dynamical systems. Lanford [42] has given computer-assisted proof of Feigenbaum's conjectures.

7.8 Part E: Dynamical system and some applications

The theory of dynamical systems palys an important role in computation and allied subjects, most notably in the area of 'analog' computation. There have been several interesting developments in this interface in recent years and this remains one of the most active areas of 'applications' of dynamical systems theory. Some of the notable topics are:

(1) *'Analog' algorithms:* Traditionally, these are continuous time, i.e., differential equation analogs of the classical algorithms for numerical analysis and optimization[43] because of the advances in analog device technology and the hope of embedding hard discrete algorithm into more tractable analog 'relaxations'. This in turn, has spawned much mathematical activity of independent interest. Two 'high points' of this trend are:

(a) *Global Newton methods:* Originally studied as schemes for computing market equilibria in mathematical economics,[44] these have attracted much attention since. A related topic is the 'homotopy' method for optimization where one tracks the global minimum of a convex function to a 'good' local minium of the function to be minimized as the former gets homotopically distorted into the latter.[45] This trajectory satisfies a differential equation similar to the global Newton method.

(b) *Brockett's double brackets:* These eqations are of the type x-[x,[s,h]] on a Lie group and originally arose out of efforts to embed discrete optimization problems into continuous flows. They are also related to Karmarkar's interior point method[46-48]. These have led to much sophisticated mathematics of independent interest[49-50].

(2) *Complexity theory for analog computation:* Computational complexity theory for discrete computation based on the Turing machine formalism is a mature subject. Efforts are on to develop a continuous counterpart.[51,52]

(3) *Neural networks*: Analog neural networks for classification problems provide interesting inverse problems. [53,54] A related activity is a study of cooperative/competitive phenomena leading to self-organization or otherwise in interesting systems of differential equations. These are of interest to evolutionary biologists, economists and engineers in addition to mathematicians.[55-57]

(4) *Control theory:* The long-standing relationship between control theory and dynamical systems theory continues unabated, with some of the more exciting developments being the use of differential geometric techniques[58] and nonsmooth analysis.[59,60]

(5) *Inverse problems:* In engineering sciences, one is not interested in chaos as such but in ways to control it. Indeed, by choosing properly the control parameters present in the systems, we wish to have a prescribed behaviour. In other words, the attractor is given and one is required to produce a suitable and meaningful system whose behaviour is described by the given attractor. [61] Another related question is the compression of data which is represented by the attractor. The attractor is, in general, difficult to describe and store. If we can get hold of the corresponding map/vf then life becomes easy. Research activities in this direction are in full swing.

(6) *Connection with nonlinear hyperbolic conservation laws:* A characteristic feature of these systems of equations is the appearance of shocks. Physically, these are limits of suitable viscous profiles as viscosity goes to zero. Finding these viscous profiles leads one to finding a heteroclinic orbit connecting the two states of a shock.[62]

7.9 Part F: Hamiltonian system

Hamiltonian systems are special DS in which the vf is given in terms of a function *H* (called Hamiltonian) defined on the manifold. Celestial mechanics provides the first examples. Since H is a constant of motion, it is natural to restrict our attention to *M*= {*H*=constant}. The natural measure on this is preserved by the flow. Under suitable hypotheses, *Poincaré recurrence theorem* then shows that almost all points on this constant energy surface are non-wandering points. Of course, one of the big classical questions is to know statistical properties of the system (Part B) : for example, whether a given system is ergodic on *M*: if not, can one obtain it at the thermodynamic limit? If so, it will justify the traditional apparatus of Gibbs ensemble in statistical physics of many particles.

In the case of Hamiltonian DS, it is customary to perturb the Hamiltonian and look for stable properties. Note that this is a restricted perturbation. Hence, we may expect new stable phenomena. Indeed, the new concept emerging is that of *elliptic equilibria* and *periodic orbits* whereas hyperbolicity is crucial in Part A. The celebrated KAM theory [63,64] studies the effect of perturbations on a completely integrable system near an elliptic point.

Quasi-periodic motions are shown to be stable depending on how irrational their frequencies are. This is a surprising result establishing some unexpected connections with number theory. The fate of rational quasi-periodic motion is described by *Poincare-Birkhoff theorem,* The break into 'island chains' with elliptic and hyperbolic points altenately placed. As in Part A, one can expect SIC near hyperbolic points. As the perturbation increases, Aubry and Mather have shown that even the irrational quasi-periodic motions disintegrate.[14] In higher dimensions, there is an additional phenomenon called *Arnold diffusion.* There are many numerical experiments[8] which give a picture of possible instabilities and bifurcations.

In the context of numerical integration of Hamiltonian systems, let us mention that the usual algorithms do not work as they do not preserve the Hamiltonian nature of the system. Hence, special efficient algorithms are needed for long-term numerical studies. In this context, let us mention the Lie algebraic perturbation theory of Dragt-Finn. [65] See also Yoshida[66] and Sanz-Serna and Calvo. [67]

7.10 Conclusion

We have presented very rapidly some important phenomena occurring in dynamical systems. Various approaches to analyse them are outlined. Apart from highlighting the progress made so far, we have also pointed out the limitations of various approaches. Through this description, it is hoped to make clear the major remaining tasks to achieve furhter progress in the field. Undoubtedly infinite-dimensional dynamical systems constitue a major challenge of the future. To handle them, on one hand, various existing apporaches will have to be generalized and strengthened and on the other, new approaches have to be discovered.

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GAME THEORY

Game Theory as a distinct discipline emerged out of the collaboration of the famous mathematician John von Neumann and an economist Oskar Morgenstern in their 1944 classic *Theory of Games and Economic Behaviour* [1]. This book had a strong impact on both disciplines. Within mathematics, this was manifested by various research projects commissioned by the RAND Corpotation, besides symposia organised by mathematician Hardd Kuhn and Albert Tucker at Princeton in the early 1950s. These early developments were related to contemporaneous developments in fixed point theory, linear and nonlinear programming and linear economic models, in all of which von Neumann played an important role. The contributions of this period lay the foundations for subsequent developments, most influential of which were the work of John Nash on-noncooperative equilibrium and on bargaining, of Lloyd Shapley on co-operative games and stochastic games, and of Robert Aumann on repeated games.

Despite a few classic contributions in the 1960s and 1970s such as Tom Schelling's 1960 *Strategy of Conflict* [2] which developed concepts and applications to international relations, bargaining and arms races, or John Harsanyy's 1967-68 work on games of incomplete information - the area of game theory was relatively quiet during this period. A resurgence of research occured in the late 1970s and early 1980s, inspired by new applications in economics, political science, international relations, and biology. This was partly inspired by the 1975 work of Reinhard Selten on subgame perfect Nash equilibrium, and the 1982 work of Ariel Rubinstein on a non-cooperative model of bargaining. The effect on economics of these developments was so profound that the 1994 Nobel Prize in economics was shared by Nash, Harasanyi and Selten.

What is game theory about? It deals essentially with strategic interaction between a number of players (e.g., firms,

traders, countries, animals). In non-cooperative game theory, for instance, player i's has available a set of potential strategies or actions A_i and receives a real valued pay off π $(a_1,.....,a_n)$, where a_i ε A_i player is problem is to select an action A_i to maximize the expected value of his payoff, given his beliefs about the actions to be selected by the other players. The theory concerns the development of suitable concepts of equilibrium action tuples, and extensions of the above to dynamic and uncertain settings. More recent developments have focused on dynamic models of behaviours, based either on *learning or evolution*. For instance, in the latter π_i represents the fitness of species or strategy in a setting of Darwinian selection, which thereafter enables studying the characteristics of species that survive in the long run. A Standard reference on the subjects is [3].

There are two aspects of game theory: One is the theoretical development and the other is the application in particular with reference to the subject of economics. First, we will briefly discuss the theoretical development. Examples are the following:

1. Abraham Wald created statistical decision theory. The backbone of this theory is two-person zero-sum games.

2. John Nash extended von Neumann's minimax theorem and proved the existence of equilibrium points for bimatrix games. His proof of this result made use of fixed point results due to Kakutani. However Lemke and Howson gave a constructive proof to obtain an equilibrium point. Their constructive method of proof lead to what is now known as Linear Complementarity problem. This is a new branch in Mathematical Programming and lot of research activity is in progress at the present time. For more details see [4].

3. Another major extension of von Neumann's minimax theorem is the concept of "Stochastic Games", introduced by Lloyd Shapely. Non-zero version of Stochastic games

have potential applications in real life problems. There are a lot of unresolved theoretical problems in this area. See the book edited by T. E. S. Raghavan et al [5] to get a glimpse of this theory. Also, see [6] to know about "Repeated Games" - introduced by Aumann and Maschler.

4. von Neumann and Morgenstern had introduced the notion of stable sets for cooperative game theory and it is still not clear wihch games have stable sets. It is a rich area where a lot of theoretical work remains to be done.

Examples of areas of application to Economics are the following (for more details see [7]):

(i). Oligopolistic industries with a few dominant firms, where their strategic inter-dependence is paramount in areas such as pricing, entry, location, product choice, capacity investment, advertising, R & D, etc. This has consequences for evolving industrial policies with respect to regulations of entry, exit, mergers or pricing of industrial firms.

(ii) Trade structure and teade policy, where most conventional theories have been turned on their head, e.g., with respect to the sources of comparative advantage, the effects of scale economics and product differentiation, and the role of interventionist trade policy.

(iii) Financial market and their regulation, which includes selection of trading strategies, speculation, regulation of stock market and of insider trading, and of financial intermediaries (commercial banks, placement of capital issues).

(iv) Technology including issues of incentives of firms to invest in R & D, transfer of technology from foreign to domestic firms, and protection of intellectual property rights.

(v) Organization design and functioning, including both corporations and large private firms on the one hand and public sector units and public administration on the other. Issues include separation of ownership and control; relation between lenders, equity holders, managers and workers; methods of planning, budgeting, production, inventories, cost allocation, transfer pricing, management compensation and labour relations.

(vi) Agricultural institutions, such as sharecropping tenancy, rural credit, rural cooperatives, and effects of government policies concerning foodgrain procurement, credit provision or laws concerning property rigths.

(VII) Environmentals preservation relating especially to ways of enforcing cooperation among potential exploiters of common resources such as common lands, forstries, or fisheries.

As a consequence, virtually any area of economics now demands a minimal knowledge of game theory. Other disciplines have also been influenced, though perhaps less significantly. One is political science, where applications have included models of electoral competition, of behaviour of legislatures and of bureaucracies, and the formation and operation of political interest groups. Evolutionary biology has also been transformed following the book by John Maynard Smith [8].

In India, awareness of game theory has been restricted to a few institutions such as the Delhi School of Economics, the Indian Statistical Institute, Jawaharlal Nehru University and the Indian Institute of Science. The large majority of students of economics and political science in Indian universities have therefore received almost no exposure of modern developments in their own disciplines. Syllabi in most universities are still based on the pre-1970 version of economics, as most of the teachers and students are unable to read new books or journals owing to their ignorance of game theory. The effect on research is obvious : Indian unversities lag behind overseas ones even more than before owing to this handicap. What is required is stronger collaboration between mathematicians and social scientists. An obvious success story in this respect is Israel, where close contact between the two disciplines has resulted in pioneering research in game theory and economics.

Thrust efforts are needed in three broad directions : (a) disseminating advances in game theory and applications to social sciences to teachers and students throughout the country; (b) promoting research in game theory, especially by enhancing collaboration between mathematicians and social scientists.

Specific suggestions for achieving these objectives are the following:

1. Summer or Winter Schools lasting a few weeks, with a target audience of college teachers, graduate and undergraduate students in mathematics, economics, political science, statistics, biology, operations research and engineering. Such courses could comprise (i) Non-cooperative game theory (ii) Cooperative and Stochastic games (iii) Applications in economics, political science and biology.

2. Lectures by game theorists in various Indian universities and research institutes.

3. Funding of research proposals and of conferences and workshops on game theory and applications.

4. Attempts to promote collaboration between mathematicians and social scientists, e.g., through workshops where mathematicians lecture on relevant areas of mathematics (measure theory, stochastic processes, graph theory, etc.) and economists present problems which require advanced mathematical tools; through programs that seek to interest mathematics, economics, social sciences students to select game theory as a research area.

5. Funds allowing universities and colleges to acquire books and journals in game theory and its applications for their libraries.

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COMBINATORIAL OPTIMIZATION

Most of us are familiar with many important practical and theoretical problems which are concerned with the choice of a "best" configuration or a set of parameters to achieve some goal. The "configuration" or a set of parameters can be formulated as variables and the resulting optimization problem seeks to find a solution that meets the constraints imposed naturally on the variables and maximises or minimises a criterion which is formulated as a function of the variables. Such problems seem to divide naturally into two categories : those with continuous variables and those with *discrete variables*. The problems with discrete variables may seek to find a combination, a permutation, a subset of a discrete set or a solution in integer values of the variables. Such problems are called *Combinatorial Optimization Problems*.

A well known example of a combinatorial optimization problem is the travelling salesman problem. A travelling salesman has to visit a given set of n cities and return to the city he started from (to be called the home city). The cost of going from city i to city j is given for each pair (i,j). The problem is to find an optimal tour, where a tour is a possible route for the salesman that covers each of the n cities exactly once and returns to the home city. It is obvious that the optimal cost is not going to be affected by declaring any particular city as the home city. Thus a tour is a cyclic permutation of the numbers {1,2,.....,n} and one seeks to find least cost cyclic permutation. Another example of a combinatorial optmization problem is the problem of finding an optimal assignment of machines to operators. Given n machines and n operators, let t_{ij} denote the time taken by the i-th operater to complete his job on the j-th machine. The problem seeks to find an assignment of machines to the operators that minimises either the total time taken by all the operators or the maximum time taken for the completion of all jobs. The obvious constraints are that each machine has to be assigned to some operator and each operator has to be assigned to some machine.

As is clear from the above, it is easy to state a combinatorial otimization problem. Also since the set of *feasible solutions* is a finite set it is easy to present an algorithm, i.e., a procedure for finding a solution. A systematic total enumeration of all feasible solutions and comparison of the criterion function evaluated at each

solution provides one with a procedure to solve the problem. However, as one can see easily, this is an unsatisfactory algorithm because with even a moderate size of the problem, the number of feasible solutions to be enumerated and compared is very high. For example, for the travelling salesman problem a measure of the size of the problem, is n, the number of cities. When n=11, the number of feasible tours is 10! Even with the development of modern computer facilities, combinatorial optmization problems remain compulationlly hard. Theoretical researches in the area of Computer Science have introduced the notion of *polynomial solvability* of problems. A problem is said to be in the class P if one can devise an algorithm to solve it so that the number of computational steps in the worst instances of the problem grows as a polynomial function of the problem size. The notion of *Nondeterministic Polynomial solvability (NP-hard)* has also been introduced. The assignment problem is, for instance, known to be an easy problem, as it is in the class P, whereas the travelling salesman problem is known to be NP-hard. Many of the network flow problems are in the class P, but there are also a large number of problems which are known to be NP-hard. A very important unsolved problem in this area is to decide if P=NP. The issues posed in this area are similar to *Hilbert's tenth problem and are very challenging.*

The major research concerns in this area include the issue of *representation*. Depending on how the problem is represented, it may be possible to devise an efficient algorithm to solve it. Very little is known about how to choose a suitable representation that will yield an efficient algorithm for a given problem. Another very practical concern is to devise *heuristic solution procedures* that will yield an approximate solution in a few iterations although, to find an exact optimal solution, a very large number of iterations may be required. Another research concern is to develop *probabilistic algorithms*.

There are many applications of combinatorial optimization in the Industries, Business and Commerce. For example, the travelling salesman problem finds an application in deciding how to cut wall paper from a very long roll of stock with a pattern that repeats at given lengths of intervals. Another application of the travelling salesman problem is to the problem of sequencing a single state variable machine. Other interesting and applicable combinatorial optimization problems include Job Sequencing, Optimal Matchings, Vehicle Routing and Optimization Problems in the setting of a graph or a network.

Research in the area of combinatorial optimization is usually done in the universities and institutes by the departments of Operations Research, Computer Science, Mathematics and Statistics. In India we have the following institutes/departments which have teaching or research interests in the area of combinatorial optimization.

1. There are groups with research interests in Network flow, Sequencing and Graph Theory at the various I.I.T's and I.S.I. (Calcutta & Delhi)
2. There are researchers working on Matching problems and computational complexity in the I.I.M.s and in I.S.I. (Calcutta)
3. The Indian Institute of Science, Bangalore, Chennai Mathematical Institute, Institute of Mathematical Sciences, Chennai and I.I.T.s have research groups with interest in computational complexity and combinatorial optimization.

The following are some of the books published recently on this subject.

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SPECTRAL AND INVERSE SPECTRAL THEORY

Many natural phenomena are modelled by some differential equations (ordinary or partial) in an appropriate configuration space. As examples one can mention (i) vibration of strings or bulk matter, (ii) propagation of sound or light and (iii) the SchrÖdinger equation for motion of an electron in an atom. In all these cases one can conveniently think of these differential operators, as operators, say A, in a suitable topological space (most often a Hilbert or Banach space).

The non-zero solutions (called eigenvectors or eigenfunctions) x_λ of linear operator equations (eigenvalue equation for eigenvalue λ) $A_\lambda = \lambda x_\lambda$, $\lambda \in C$ are of special interest. In the context of the time-evolution of the system under consideration, they describe the stationary solution and it is of great interest to have a complete description of the possible eigenvalues λ and corresponding eigenvectors x_λ. Even in cases when the differential equations modelling the natural phenomensa is non-linear, (e.g., for general motion in classical mechanics, for fluid flow in fluid mechanics or propagation of light in non-linear optics), one studies the local classification problem for the associated dynamical system by linearizing the equation and looking at the eigenvalues of the linear differential operator (*see the write-up on Dynamical Systems)*.

From a pure mathematical point of view it is useful to start the study of the problem in finite dimensional spaces and this is a well understood area (see e.g. [5]). After this a natural question that arises is that of perturbation and stability of eigenvalues and eigenvectors. This is of considerable importance also in applications in atomic and molecular physics and chemistry. One considers an eigenvalue equation: $A(z)x(z)=\lambda(z)x(z)$ where z may vary in some region of the complex plane containing 0 and $z \rightarrow A(z)$ is a continuous (or holomorphic) matrix valued function with $A(0)= A$ and wants to study the dependence of the eigenvalues $\lambda(z)$ and the eigenvectors on z. A lot of results are known (see, e.g., [5] and [3]) but some subtler questions still remain to be answered. Many of these results are also of considerable interest in numerical analysis for example in the analysis of various algorithms for solving systems of equations and in error analysis in computations.

An identical set of questions can be posed in an infinite dimensional space. Immediately the problem takes a different dimension, viz., one has to allow also unbounded operators (sometimes a manageable subclass of self-adjoint operators) as well as the possibility of having a 'kind of continuum of eigen-values', i.e., continuous spectrum in contrast to point spectrum. If the unbounded operator is not normal (or self-adjoint), then there is the additional complexity of classification of its spectrum and associated eigenspaces. For the self-adjoint case in general and the two-body Schrödinger (partial differential) operator, the problem is fairly well understood (see [5], [6], [2]). But for n-body Schrödinger operators (which are of great interest in real atomic or morecular physics), there are many questions still to be answered.

The perturbation or stability question in infinite dimension is similar to that in finite dimension, but now one has to admit the possibility of a proper eigenvalue 'melting into the continuum' as a result of perturbation or conversely eigenvalues emerging out of the continuum. The first kind is often used in modelling resonance in atomic or nuclear physics while the second is unavoidable in n-body Schrodinger systems and can lead to the so-called Effimov effect. The problem of resonace, in particular, needs further study and is one of the many factors responsible for the broadening of the atomic and molecular spectral lines as well as for the decay of unstable systems (see also write-up on *Stochastic Modelling)*.

Very little is known about the spectrum of non-normal unbounded operators and their pertubations. However, one has some results on the perturbation of compact (non-normal) operators and their properties w.r.t. various Schatten-norms as well as the properties of the associated perturbation determinants (see e.g., [3], [4]). This is an area which has seen considerable activity in recent years and it is expected to continue in near future.

Often the eigenvalue problem of the unperturbed operator (e.g., the case of free Hamiltonian of quantum system) is relatively simple. Then one tries to relate the eigenfunction of the perturbed operator to that of the unperturbed one; this can take the shape of an inhomogeneous operator equation or in the concrete case of Schrödinger operators, integral equation in configuration space. This forms the subject of eigenfunction expansion (2).

Another approach is that of scattering theory in which one studies how a one-parameter unitary group behaves under perturbations of its generator, and tries to construct wave operators which intertwine the two groups. With the help of these intertwiners, one can easily describe the various spectral subspaces of the perturbed operator if one knows these for the unperturbed one. This also leads to the study of S-matrix which is of central importance in atomic and nuclear physics (6), (2).

One can turn the eigenvalues or spectral problem upside down and ask. Given the spectral data (spectrum and spectral density), can one construct the differential operator whose spectral data are precisely those given? This is the subject of Inverse Spectral Theory and has seen major activity in the last decade after the pioneering work of Gelfand, Levitan and Marchenko long time back.

Another area which has been in the lime light is the spectral and inverse spectral theory of the random Schrödinger operator or its discretized version, viz., (infinite) random Jacobi matrices. Here the potential or perturbation is a multiplication operator-valued random variable in $L^2(R_n)$ or $l_2(Z^n)$ while the unperturbed operator is either the Laplacian or the second order difference operator (Jacobi matrix). The random potential is supposed to model the presence or introduction of impurities in a crystalline solid and the electronic spectral structure changes considerably. In one and two dimensions, one can show that in most situations, the eigen function is exponentially localized in space "C Anderson localization" for almost all values of the random parameter and gives rise to absence of electrical conductivity.

The broad field of operator algebras (C*-and von Neumann algebras) have seen spectacular developments in the last couple of decades, both as a pure mathematical discipline as well as a tool for applications in diverse areas like quantum mechanics [2],

topological field theory, quantum stochastic processes (see also the *write-up on Stochastic Process Modelling)*, etc. Most fascinating has been the very recent developments in non-commutative topology and geometry [1], in which topological and geometrical ideas in classical spaces are lifted to associated algebraic properties on the commutative algebras of functions on the spaces and then the commutative structure (or equivalently the underlying space) is thrown away. Using these techniques, there have been successful studies of the spectra of the Laplacian and the Dirac operators on manifolds and of their associated index theorems.

There are a few institutions in India where research work of international standards in these areas has already been done, e.g., the two centres of the Indian Statistical Institute at Delhi and Bangalore, and the Institute of Mathematical Sciences at Chennai. Work on numerical aspects of the eigenvalue problem is being pursued at I.I.T. Mumbai. Work in other areas of Functional Analysis and operator theory is also being done in the departments of mathematics of the Universities of Delhi, Pune, Mumbai, Punjab, Jammu and at Sardar Patel University at Vallabh Vidyanagar. A few workshops have been held in the past in some of these areas sometimes under the sponsorship of the Indian Academy of Sciences, Bangalore. But much more needs to be done to develop manpower in these areas-more training workshops, encouraging visitors' programme in the departments of mathematics of a few selected universities including visits of a few eminent experts in these areas from abroad, are a few things one can try.

As has been mentioned earlier, these areas of mathematics have many applications in Physics and Chemistry and more indirectly in Engineering Sciences. Thus programmes and projects of interdisciplinary nature involving these disciplines may be encouraged particularly between a few selected universities and national institutes. Since there is a serious resource crunch, it is felt that it will be more effective to support intensively a very few selected good departments/institutions with strong track records in these areas instead of making much larger investment of building a new set-up.

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WAVELET ANALYSIS

11.1 Introduction

The objective of this write-up is to highlight the importance of wavelets in the study of Partial Differential Equations (PDE) and the Numerical Analysis (NA) for their solutions. The use of Fourier Analysis (FA) in this domain is classically well-known in the form of Fourier Transform (F.T.), Fourier Series (F.S.) and Spectral Methods (SM). Other techniques which are frequently used in the numerical computations include the Finite Element Method (FEM) and the Finite Difference Method (FDM). If seems natural

to begin by recalling some of the virtues of these classical methods and the difficulties that we face in enlarging their field of applications. Perhaps, this is the best way of motivating the definition of wavelets.

National and international conferences and popular lectures organized in the last few years show the enormous interest of the scientific community towards the subject. At the same time, we witness an explosion of articles and publications in the journals presenting seductive properties of wavelets and their applications in various domains such as harmonic analysis, numerical analysis, computing, image processing, signal processing, fluid mechanics, etc. There even exists an upto date bibliography on wavelets available through e-mail at nominal cost. Considering this situation, one feels the need to stimulate interest and develop the subject in India. Of course, certain individuals in India, realizing the importance of the subject, are already pursuing research in this attractive field. Our aim here is to achieve this by probing the following questions: What are wavelets? Why wavelets? What are their properties? Why are they better suited than their predecessors to understand various classical phenomena in a different light? What new things can be achieved using them? As we shall see, they cannot replace FA; on the contrary, we need FA to understand and construct wavelets. We touch upon certain applications of wavelets, and conclude by discussing recent developments, modifications, improvements, various perspectives and an outlook into the future.

11.2 Fourier Analysis

Since the intial ideas of Fourier, trignometric series and the Fourier Transform (F.T.) have been the main tools to study the structure and regularity properties of functions. The definition of F.S. and F.T. of a function f defined on R stems from our desire to represent f in terms of the exponentials $\{e^{i\xi x}\}$. Since the latter functions are "nice", we expect to read off “understand” the properties off.

One crucial question is the following : Do we loose any information in passing from f to its F.T. ^ This is not easy to answer. It depends on the class to which f belongs. If $f \in L^2$, then we do not loose any information. In fact one can read off whether or not a function $f \in L^2$ by merely looking at the magnitudes of its Fourier coefficients. The same is true of the Sobolev spaces H^s which are based on L^2. Further, the convergence in the inversion formula takes place in the corresponding norm. The significance of these spaces is that their elements represent states of several mechanical systems with finite energy. To analyze finer properties of such systems, in particular to study nonlinear systems, we need to consider L^p, C^0 spaces and more generally $W^{s,p}$, C^s space. We are then naturally led to ask the following questions:

- Can one characterize f ε $W^{s,p}$, (fε C^s) in terms of the absolute values of f?

- Does the convergence in the inversion formulae take place in the corresponding norm?

The answers to these types of questions are in general difficult and negative. The reason is that $\hat{f}$ has a phase even if $\hat{f}$ does not have one and these phases play a role which is too subtle to be unravelled by human beings.

On the other hand, let us recall the following striking property of F.T. with respect to differentiation which has enormous success in linear PDEs: The analytic operation "derivation" converts into the algebriac operation "multiplication by a polynomial" under F.T. This is because exponentials are eigenfunctions of constant coefficient operators. These properties lie at the heart of the analysis of linear PDEs with constant coefficients. The case of operators with smooth variable coefficients is harder. However since there is no major change qualitatively, the problem can be attacked by perturbation analysis and this requires sophisticated tools such as

the calculus of Pseudo Differential Operators and Fourier Integral Operators. However these methods are not easily adaptable to cover nonlinear equations which are the order of the day. This is because exponentials are no more left invariant and this is a qualitative departure from the linear case. This simple reason is good enough to look for alternatives of F.A.

In the theory of PDE's one is interested in the following aspects of solutions apart from their existence and uniqueness:

- regularity properties with respect to data,
- singularities if any,
- position, size and nature of singularities given these data initially.

To analyze the singularities of f, the so-called singular support of f is introduced. In order to keep track of the singularities, one discovers that one has to consider also the corresponding wave numbers which cause the singularities. This localization in (x,ξ) space leads us to one of the fundamental objects, WF(f), called the *wave front set* of f. Comparing the singularities with the energy of a wave front of light, we conclude that WF(f) obeys the laws of *Linear Geometrical Optics* (LGO) in the case of linear PDEs. For reasons given above, the analysis of WF(*f*) for nonlinear equations poses a great challenge to mathematicians. As we shall see, localization in the physical as well as fourier space lies at the heart of wavelet Analysis (W.A). Indeed WF(f) should be compared with the set of points where the wavelet coefficients $W_0(b,a)$ do not "vanish".

If we examine the difficulties mentioned above a little more closely, we see that one of the principal difficulties is that exponentials are localized to the maximum in ξ-space. They are very regular and have no decay at all. According to the *Heisenberg Uncertainity Principle,* the more an object is localized in ξ-space the more difficult it is to describe the local phenomena in x-space.

This explains why we face serious difficulties in describing regularity properties of function f using its Fourier representation. Can one replace exponentials by other functions which do not concentrate in ξ-space and have nice decay properties in x-space? Can they be constructed by easy means? Do they form a basis? The definition of wavelets is motivated through these questions.

Some of these questions were earlier asked in the context of generalizing LGO to nonlinear equations.This is the subject matter of *Nonlinear Geometrical Optics* (NGO). The crucial idea there is to superimpose exponentials over several wave numbers to obtain a suitable localization in x-space. In other words, replace exponetials by a suitable function which will be determined in such a way that it has some desired properties. Some of these ideas are retained in the construction of wavelets also; however the desired properties are not the same now.

11.3 Haar Bases

If the difficulty with exponentials is what was described above and the purpose is to describe local properties of functions then one obvious solution is to look for a basis whose elements i.e., functions are localized in x-space rather than in ξ-space. The construction of the classical Haar basis is done with this in mind. It forms an orthonormal basis for L^2 (R). Compared to exponentials, the Haar basis have many advantages. For instance, the norm of an L^p function can be estimated by a function depending only on the absolute values of Haar coefficients of f. Nothing similar could happen with exponentials.

One of the drawbacks of Haar functions is that they fail to be eigen functions of constant coefficient operators. They are not differentiable at all. This failure is due to the fact that Haar functions are on the other extreme; they are much too localized in x-space and poorly localized in ξ-space. This is reflected in their lack of regularity and oscillations.

The moral therefore is that we should not completely sacrifice the localization in ξ-space and the oscillation property available in F.A. even though there is a need to localize x-space. So, the idea is to strike a middle ground between these two extremes without violating *Heisenberg's Uncertainity Principle* but touching the very limit setforth by it. Wavelets arise naturally in this way.

Another idea tried out in the past is to smoothen the Haar basis by taking their primitives, but then one looses the orthogonality property. By the classical Gram-Schmidt orthogonalization, one can recover it but then the functions obtained this way introduce enormous computational complexities. Recall that the computation of solutions is one of our principal aims. Complexity means a lot of operations in the computer and thereby increase in the cost and round-off error. Complexity is thus to be avoided.

11.4 Numerical Analysis

Having seen some motivation for wavelets from the Fourier analysis of solutions of PDEs, we turn our attention to the computational aspects and point out some fundamental difficulties. What can be done to overcome them? As we shall see, this leads us to wavelets once again.

The usual procedures employed to discretize PDEs are FDM, FEM and S.M. The basic idea is to approximate the spaces involved by finite dimensional spaces. To construct them, exponentials are used in S.M. whereas piece-wise polynomials are used in FEM. The question is : How accurate are the approximate solutions ? Apart from the order of the scheme, this is related to the regularity of the solution and the stability. In classical situations where the solution is regular enough, the error is of finite polynomial order in FEM and of infinite order in S.M. However, in situations where the solution f is not regular, spectral approximations do not yield satisfactory results. One such example is the velocity field of a turbulent flow. Its principal characteristic is that its Fourier representation is"full", i.e., there exists N large such that $|\xi|>N$ are all neglible and $/\xi/\leq N$ are not neglible. Hence it is intuitively clear that if we want a reasonable approximation of such functions we must take into account all Fourier modes! $/\hat{f}(\xi)!$. The limitation of today's computers in terms of memory requirements and the speed of calculations prevent us from doing this. Rigorous mathematical analysis of these solutions are out of reach for the moment. The idea therefore is to look for alternatve basis functions in which solutions will have "controllable" number of terms which are significant. Once again, we see the need to superimpose exponentials over several wave numbers. Wavelet representation is motivated towards carrying out this idea.

The situation with FEM and FDM is not bad. On the one hand, the F.E. basis functions are easily constructed even on unstructured grid avoiding complexities. On the other hand, there are some adhoc procedures to handle singularities. From physical reasons, the location of singularities of solutions is roughly estimated. For turbulent solutions, this is a hard problem and there are only conjectures. Once this is done, refinement of the mesh in those regions is performed. This amounts to a minimal increase in the dimension, guaranteeing, at the same time, an enhanced accuracy of the approximate solution.

This practice has been in existence for quite sometime with the numerical analysts and it is found quite successful. In some cases, there has been mathematical justification. As we shall see below, the introduction of wavelets is to formalize this adhoc procedure. The F.E. basis associated with such meshes are, of course, localized in x-space but non-uniformly distributed in space to take care of the variation of functions. Their main drawback is that they are not very smooth, neither do they have the oscillation property mentioned earlier. Hence it is necessary to combine this basis with that in S.M. in a suitable sense.

11.5 Wavelet Transform

From our discussion in the previous sections, we feel the need to have a basis consisting of functions localized in (x,ξ) space. The notion of WF (f) already incorporates such an idea. Another classical object which does the same job is the *Windowed Fourier Transform* (WFT) introduced by GABOR. The idea is to$_{(t)}$ decompose the given function into small pieces (windows) and take F. T. of each piece. More precisely, WFT (t) of a function (fx) is defined by

$$Tf(\xi,x)=\int f(y)g(y-x)e^{i\xi y}\, dy,$$

Where g is the fixed window function. One of the drawbacks of this localization is that regardless of the frequency values (high or low) the windows have the same width defined by g. Intuitively, we feel the need for larger windows to see high frequencies and smaller windows to see low frequencies. The definition of Wavelet Transform (W.T.) can be seen to achieve this. A second reason to modify WFT is that it has been shown that one can only generate "frames" and not a basis via a lattice sampling in WFT. On the other hand, as we shall see in the sequel, it is a miracle that a suitable lattice sampling of W.T. will lead us to an orthonormal basis. The formalization of the above ideas involve the following: Since we wish to localize in x-space, we must have a variable to do this job. Since exponentials were localized in ξ-space, this was not possible in F. T. Of course as in F.T., we must have a variable which measures the scale of variations of functions. As agreed upon already, exponentials have to be grouped over several wave numbers and this gives rise to what is called a *mother wavelet* function ψ. Once Ψ is choosen, the principale of *Wavelet Transform* (W.T.) is very simple. As in F.T., given a function f, we test it against Ψ. Let $b\in R$ denote the position parameter which can be moved from one position to another by translation. This corresponds to localization in x-space. Let $a>o$ be the scaling parameter which measures the scale of variations of functions. This corresponds therefore to localization in ξ-space. They form a group under multiplication. W.T. of f is defined as follows:

$$Wf\,(b,a) = a^{-1/2} \int f(x)\ \psi\ (^{x-b})\ dx.$$

It is quite clear that W.T. serves as a "mathematical microscope" to analyze the structure of a function. Indeed, by fixing b, we can localize the behaviour of f around b, and by decreasing the values of $a > o$ we can see the structure of f to finer details.

Of course, the mother wavelet ψ is not an arbitrary function. From our discussions in the earlier section, we wish ψ to have the following properties:

(a) ψ is regular,

(b) ψ is localized in x-space,

(c) ψ oscillates,

(d) there exists an inversion formula expressing f in terms of Wf.

A partial solution to the above question was found in the early 60"s by CALDERON.

The single most important property of W.T. which distinguishes it from F.T. is the following : f is regular at b if the wavelet coefficients Wf(b,a) decay as $a \to 0$.

The better localization of Ψ in x-space the poorer is its localization in ξ-space and, in this case, the properties which are due to the maximum localization in ξ-space are more violated by W.T.

11.6 Wavelet Series

A priori the uncertainity principle in Quantum Mechanics seems to cast doubts on obtaining a basis consisting of localized functions in (x,ξ) space. However there is one such basis. From the numerical point of view, this is extremely important because it implies enormous reduction in the storage.

To see this, let us start with the classical finite element spaces which are defined on finer and finer meshes of R: $V_j=\{f\in C^o(R)$; f is linear on $[k2^{-j}, \{k+1)2^{-j}]\ \forall\ k\in Z\}$, $j\in Z$. These spaces correspond to the so-called p_j-element in the finite element literature. The usual finite element basis consisting of "hat functions" has a particular structure which was not used in the theory of finite elements but becomes important now. It is the following: forV_0, there exists a basis of the form $\{g(x-k)\}, k\in Z$, where $g \in V_0$. In fact g is the unique hat function in V_0 such that g (0) =1 and g(k) = 0, $\forall k\in Z\backslash(0)$.
The above example can be abstracted and this gives rise to the concept of Multiresolution Analysis (MRA) which in turn implies the existence of a mother wavelet ψ and a father wavelet φ.

Once their existence has been established, the next concern is their suitability for a local Fourier analysis. So, can one choose ψ such that ψ is regular, ψ is localized in x-space and ψ has oscillation property? In this connection, the following results have proved in the literature STROMBERG proved the existence of $\psi \in C^s$ having exponential decay and oscillations. DAUBECHIES improved it by showing that ψ can be chosen to have compact support. On the other hand, LEMARIE & MEYER showed that the choice of an oscillating ψ is possible in the Schwartz class.

11.7 Wavelet series and Fourier Series

W.S. are destined to compete with F.S. Thanks to the double localization, W.S. permits us to do analysis, finer than F.S as far as

W.A. is concerned, the single property which most distinguishes W.S. from F.S. is the following; W.S. is "sparse" in the sense that the wavelet coefficients with respect to scale parameter j are "zero" for j large where the function is regular. This property is responsible for enormous data compression. On the other hand, let us remark that Fourier series of important functions are "full" whereas lacunary F.S. represent often pathological functions.

Moreover, wavelets provide a basis also for the classical standard spaces L^p, $1 < p < \infty$, $C^{o\alpha}$, $0 < \alpha < 1$, etc. These spaces are characterized directly by conditions on the wavelet coefficients. Let us remember that such characterization with Fourier coefficients are rather rare.

These are, of course, certain inconveniences in dealing with wavelets of which we mention two here. Recall that the derivation operator is transformed to the multiplication operator under F.A. This property is no more true. However, some operators acquire special structure in the wavelet formulation depending upon the choice of the wavelet. For instance, the so-called Calderon-Zygmund Operators are diagonalizable in the wavelet basis.

Next, turning our attention to nonlinear equations, let us recall that the usual multiplication in x-space is transformed to convolution product under F.A. In other words, the Fourier coefficients of f2 are calculable entirely in terms of those of f. This is not the case with the wavelet coefficents. For the moment, this is done in x-space after computing f^2. Research is on as to how best the wavelet coefficients of nonlinear terms can be directly calculated without going to the physical space.

11.8 Wavelets in Numerical computations

If there is one field where wavelets have an enormous impact it is in the domain of numerics. Since a wavelet basis lies

between a finite element basis and a spectral basis as explained already, it shares their properties : It is as efficient as FEM in localizing and capturing singularities of solution and at the same time providing good approximation in smooth regions. This latter phenomenon depends on the oscillation property satisfied by the wavelets.

A major task is to exploit the presence of lacunarity in the wavelet series representing the solution. To this end, we must use necessarily non-uniform meshes. Indeed, comparative study shows that on regular meshes, the wavelet method and more traditional methods yield the same type of results. In practice, the mesh is rendered non-uniform in an iterative fashion by anticipating the significant wavelet conefficients at the next iteration from the magnitude of the coefficients in the present iteration. Another technique is to use what are called *mobile wavelets.* The idea here is to consider the wavelets as particles in the space (b,a) of position and scale and they move around as time evolves. The approximate solution is in the from

$$\sum_{i=i}^{n} c_i(t)\, \psi\left(\frac{(x-b_i(t)}{a_i(t)}\right)$$

The aim here is to cook up suitable evolution equations for a_i (t), b_i (T) and c_i(t) in such a way that there is a strong concentration of wavelets in the region of the singularities of the solution.

Yet another theory on the horizon to achieve this is that of *Wavelet Packets* wherein the aim is to represent a function on a basis which is optimal in the sense that the number of elements of the basis representing the function is as small as possible. Each element of the basis is constructed starting from the mother-wavelet packet by the operation of dilation, translation and modulation. Thus there are three parameters instead of the usual two in the classical construction of wavelets. This theory includes that of windowed F.T. and W.T. and seems to be full of promise in future applicantions.

11.9 Conclusion

In this write-up, we have tried to answer the following-questions: What are wavelets? Why wavelets? Basic ideas behind their construction with examples and interpretaions starting from the classically known object, their immediate porperties, their limitations, comparison with trigonometric functions, etc., have also been dicussed. There are several issues which are not discussed. To the topics covered by several questions raised in the ealier sections, we add the following ones: The choice of wavelets best suited to the problems at hand, construction of wavelets in the presence of boundaries, issues involved in the case of several variables, wavelets in nonhomogeneous media, etc. Wavelets are destined to compete with the more classical trigonometric functions. Consequently, some of the classical things are viewed in a different light now. For instance, Calderon-Zygmund operators are almost diagonalizable in wavelet basis and this explains a *posteriori* their success. Various algorithms using wavelets for these operators should show rapid convergence and this has to be confirmed.

A basis goal of the subject is to analyze the singularities of solutions of nonlinear equations. One specific question in this context is the following: Do Navier-Stokes equations and Euler equations in three dimensions exhibit non-smooth solutions with smooth initial data? Can one answer such questions using wavelets? F.T. did not have much success in this area. There is numerical evidence of an affirmative answer to this question. However, rigorous mathematical analysis of this phenomenon is still elusive. The conjecture is that the set of sigularities is concentrated on a small set. If this is true then, by the very virtue of wavelet coefficients, we will be able to represent fluid flows by wavelet series where there exists only a "controllable" number of significant

terms. Fourier analysis enabled one to derive upper estimates on the dimension of the attractor which represents the fluid flow. The above arguments may imply that a significant improvement of this estimate can be achieved using wavelets. Can one then develop a nonlinear Galerkin method based on wavelets? Probably, these are some of the major issues with which the scientific community will be preoccupied in the future.

As mentioned already, there is not much expertise available in India. On the other hand, thes are strong research groups working in Fourier Analysis (ISI, TIFR-Bangalore, IIT-Kanpur, etc). It is therefore suggested to group the interested persons in these organizations and with their assistance, organize a series of meetings, schools, workshops, etc. on various developments in Wavelets. Hopefully this may stimulate interest in the field among a wider section of the community.

As stated at the begining, there is an ever growing list of works in this rapidly growing area. The bibliography here is just a representative sample and includes works cited in the article.

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